Algorithms for Real Numerical Varieties with application to QSIC

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1 Introduction

In this paper I outline some algorithms for working with real numerical varieties with an application to parameterizing quadratic surface intersection curves (QSIC). A revised version of this paper has now been published [8].

Although possibly the simplest space curves, the study of QSIC has been the subject of recent research [9, 15, 16]. From the point of view of parameterizing QSIC the papers of L. Dupont, D. Lazard, S. Lazard and S. Petitjean [9] essentially solve the problem. Although their method requires starting with exact systems, takes 65 pages and involves looking at many cases the accompanying software [10] is extremely fast and accurate. The coefficients, which must be integers, can be quite large so can adequately approximate most numerical systems.

In this paper I describe a numerical based method. Although I doubt I could ever achieve the completeness and speed of [10] my method is straight forward and, as you will see in this report, can be described using standard methods of numerical curves, with the one step specific to QSIC described in the few pages of Section 5. Moreover this section simply reformulates a classical argument into numerics. Although, for ease of replication, examples in this report are given exactly, the method immediately switches to an equivalent numerical system, so examples could be given numerically. Unlike [9] which uses the projective line as a parameter space I use simple affine real plane curves of the form $y = u(x)$ or $y^2 = u(x)$ for a real numerical polynomial $u(x)$ of degree at most 3.

The tools used consist of (1) fractional linear transformations [1] given by a matrix based presentation, (2) Macaulay and Sylvester matrix based computations for decomposing numerical curves into irreducible components and finding equations for images of curves under polynomial maps, and (3) methods involving numerical polynomial system solving to find real points on numerical algebraic varieties. For (3) these method are quite recent and this is the first exposition of these methods.

The method for identifying and parameterizing QSICs has the following steps:

1. Do a (complex) numerical irreducible decomposition to identify algebraic components. If all components are of degree 1 and/or 2 go to Step 5.
2. Find a random real nonsingular real point, if any, of the QSIC.
3. Using the original system, find a numerical cubic plane curve birationally equivalent to a union of components of the QSIC (Main Theorem on QSIC) and the birational transformations.
4. Separate the cubic into irreducible components, use 1) to check whether all components of original QSIC with real points are accounted for. Otherwise the missing line will come from Step 1.
5. Transform each component via fractional linear transformations to parameter curves of form $y = u(x)$ or $y^2 = u(x)$.
6. Analyze the rational parameterizations on the parameter curves to obtain practical parameterizations of the QSIC.

Note this last step is not covered in [10]. The only step specific to QSICs is step 3.

In §8 below we give a complete example using this method.

The next three sections deal with general techniques. In this paper the phrase *algebraic set* will refer to the point set $\mathcal{X}$ in $\mathbb{R}^s$ of solutions of a system of real polynomials in s-variables. On the other hand *algebraic variety* will refer to an ideal $\mathcal{I}$ of $\mathbb{R}[x_1, \ldots, x_s]$ such that $\mathcal{X} = V(\mathcal{I})$.

2 H-bases and Duality Method

For numerical work the equivalalent of a Gröbner Basis is an *H-basis* [13, 14], also known as a *Macaulay basis* [12, §4.2]. An H-basis of an affine ring, eg. ring of the form $A = \mathbb{C}[x_1, \ldots, x_s]/\mathcal{I}$, is a set $\{f_1, \ldots, f_n\} \subseteq \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_s]$ such that if $f \in \mathcal{I}$ then there exist $g_1, \ldots, g_n \in \mathbb{C}[\mathbf{x}]$ such that $f = g_1 f_1 + \ldots g_n f_n$ where for each i $\deg(g_i f_i) \leq \deg(f)$. Note that if $B = \{f_1, \ldots, f_n\}$ is a homogeneous basis of $\mathcal{I}$ or is a Gröbner basis with respect to a positive degree ordering of $\mathcal{I}$ then B is an H-basis.

A theory of local-global duality is outlined in [6], a more recent summary has been given in [7].

In particular there are two numerical algorithms that I will use extensively in this paper

Algorithm 1: Given a basis B, not necessarily H-basis, for ideal $\mathcal{I} \subseteq \mathbb{R}[\mathbf{x}]$ and points $\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k \in V(\mathbb{R}[\mathbf{x}]/\mathcal{I})$ an H-basis is returned for the variety $Y = V(\mathcal{J})$ which is the union of irreducible components of $V(\mathcal{I})$ which contain one or more of the points $\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k$.

Algorithm 2: Given a real H-basis for the ideal of variety $\mathcal{X} = V(\mathcal{I})$ and an algebraic map $\phi = \{\phi_1, \ldots, \phi_s\} : \mathbb{R}^s \mapsto \mathbb{R}^p$, the $\phi_i \in \mathbb{R}[x_1, \ldots, x_p]$, a real H-Basis will be returned for the variety $V(\mathcal{J})$ which is the Zariski closure of $\phi(\mathcal{X})$.

It should be noted that these algorithms require the user to supply both a numerical tolerance and appropriate degree to assure an H-basis. If the user has his or her own favorite algorithms to do these calculations they may be substituted provided that they do work for ideals defined by floating point polynomials.

3 Fractional Linear Transformations

We will make use of fractional linear transformations [1], that is transformations of the form

$$\mathbf{x} = (x_1, \ldots, x_s) \mapsto \left(\frac{\alpha_1(\mathbf{x})}{\delta(\mathbf{x})}, \ldots, \frac{\alpha_s(\mathbf{x})}{\delta(\mathbf{x})} \right)$$

where the α_i and δ are linear functions in s variables, i.e. $\alpha_i = \alpha_{i,1}x_1 + \cdots + \alpha_{i,s}x_s + \alpha_{i,s+1}$. If A is the $(s+1) \times (s+1)$ matrix

$$A = \begin{bmatrix} \alpha_{1,1} & \ldots & \alpha_{1,s} & \alpha_{1,s+1} \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ \alpha_{s,1} & \ldots & \alpha_{1,s} & \alpha_{s,s+1} \\ \delta_1 & \ldots & \delta_s & \delta_{s+1} \end{bmatrix}$$

then this is the transformation

$$\mathbf{x} \mapsto A \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ 1 \end{bmatrix} = (y_1, \ldots, y_s, y_{s+1}) \mapsto (y_1, \ldots, y_s)/y_{s+1} \tag{1}$$

In effect we are homogenizing, applying a linear projective transformation and then specializing again. Note that if we compose the fractional linear transformation given by A with the one given by B we get the fractional linear transformation given by BA. In particular if A is invertible then the fractional linear transformation given by A is birational.

To find the action of the fractional linear transformation associated with a matrix A on an affine curve just follow the instructions in (1), homogenize, transform using Algorithm 2, and dehomogenize.

4 Finding Real Points on Curves

My main algorithm later on requires finding a random real point on the curve. The first thing to try is to intersect the curve with random real hyperplane and check for real solutions. One may repeat several times if a real point has not been found. However, in general the real

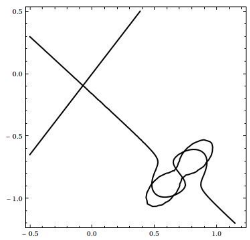


Figure 1: Real plane Curve (peanut), Jacobian Determinant curve and ℓ

locus of an algebraic set can be quite small or even a finite set so this simple method may not produce a real point.

In the case of a plane $f(x, y) = 0$ curve a very efficient way to find real points is to look for real solutions of the system $\{f, \mathcal{J}(f, \ell)\}$ where $\ell = ax + by$ for random or chosen real numbers a, b not both zero and $\mathcal{J}(f, \ell)$ is the determinant of the Jacobian of $\{f, \ell\}$. This idea was motivated by the paper [4]. The picture in Figure 1 shows how $\mathcal{J}(f, \ell)$ grabs the real locus of the curve. The points found include all those where the tangent line to the curve is parallel to the line ℓ as well as any singular points, including any isolated points of the curve. If the curve has an *oval*, which means in this paper a non-empty bounded topological component, then at least one real point will be found. Further, letting $\ell = x$ or $\ell = y$ will find the x, y bounds of the oval.

For space curves, or curves in $\mathbb{R}^s$ in general, we can try the following based on the paper [3]. Let $F = \{f_1, \ldots, f_{s-1}\} \subseteq \mathbb{R}[x_1, \ldots, x_s]$ define a curve in $\mathbb{R}^s$. Pick a non-zero linear form $k = k_1x_1 + \cdots + k_sx_s$ randomly or purposefully.

Let $J(f_1, \ldots, f_s, k)$ be the Jacobian matrix, B a $s\times s$ orthogonal matrix, and $C = [1, c_1, \ldots c_{s-1}]$ a $1\times s$ matrix where the c_i are new variables. Then $A = C.B.J$ is a $1\times s$ matrix $A = [a_1, \ldots, a_s]$, let $G = \{a_1, \ldots, a_s, f_1, \ldots, f_{s-1}\}$. Then G is a system of $2s-1$ equations in the $2s-1$ variables $x_1, \ldots, x_s, c_1, \ldots, c_{s-1}$. Find the real solutions for this square system G, and discard the last $s-1$ coordinates corresponding to the $c's$. If s is small and the system is exact one could use Gröbner bases with an elimination order to get an $s \times s$ system. The solution points in R^s should be the points where the curve intersects a hyperplane parallel to $V(k)$ singularly, in particular isolated points should be found. If the curve is an oval any k should be successful otherwise different k should be tried. Failure of this method may be caused by incomplete identification of the real solutions of G. Since this can be a common problem of solvers which find all complex solutions, what makes this method appealing is that we obtain a square system which is more likely to properly suggest real solutions than when trying to solve over or under-determined systems.

Example: (Example 3 of §9 below). Consider the QSIC $V(\{f, g\})$ given by $f = x^2 + z^2 - 2y,\ g = -3x^2 + y^2 - z^2$. This space curve and the birationally equivalent plane curve h of Example 3 both have isolated real points. To find an isolated zero of h we choose, randomly, $l = 0.816353x - 0.273704y$ and solve the system $\{\mathcal{J}(h, l), h\}$ getting 6 real solutions of which 2 are the multiple zero $(0.103102, -0.0989506)$ indicating a singular solution. Further inspection reveals that this is an isolated zero.

For the space curve $V(\{f,g\})$ we pick a random real linear form $k = -0.668293x - 0.286001y - 0.214335z$ and form $A = C.B.J$ where B is a random orthogonal matrix and J is the Jacobian matrix of the system $\{f, g, k\}$

$$A = [1, c, d] \begin{bmatrix} -0.658953 & -0.557442 & -0.505014 \\ 0.70223 & -0.696511 & -0.147464 \\ -0.269545 & -0.451808 & 0.850421 \end{bmatrix} \begin{bmatrix} 2x & -2 & 2z \\ -6x & 2y & -2z \\ 0.816353 & -0.273704 & 0 \end{bmatrix}$$
$$= [-0.41227 - 0.120382c + 0.694244d + 2.02675x + 5.58353cx + 2.17175dx,$$
$$1.45613 - 1.3641c + 0.306327d - 1.11488y - 1.39302cy - 0.903615dy,$$
$$- 0.20302z + 2.79748cz + 0.364524dz]$$

Solving the system consisting of the three polynomials above in x, y, z, c, d and f, g by Mathematica and Bertini 4 real solutions are identified by the two solvers which give essentially the same results. Two of these zeros essentially agree and give a multiple solution with $(x, y, z) = (0, 0, 0)$ indicating a singular point which is, in fact, isolated. The two other solutions for x, y, z are $(\pm 3.4641, 6.0000, 0)$, giving random real points on the 1 dimensional real component of the solution.

I should mention that discussion of this example with Daniel Lichtblau motivated the material in this section.

5 Main Reduction

Here we give a constructive proof of a numerical simplification of a classically known fact: a QSIC is generically birationally equivalent to a degree 3 plane curve plus, perhaps, a line.

Before stating this formally we recall Abhyankar [1, p. 21]

> A theorem is not something that is true, but is rather a nice geometric statement that you want to be true. So you adjust your definitions properly

To this end we define a *real affine QSIC* to be a QSIC $V(\{f,g\})$ such that every complex projective component has a real positive dimensional affine solution set.

Theorem 1 *Let $\mathcal{C}$ be a real affine QSIC. There is a plane cubic h and rational maps $\Phi : V(h) \to \mathcal{C}$, $\Psi : \mathcal{C} \to V(h)$ such that $\Psi \circ \Phi = id_{V(h)}$. In particular $V(h)$ is birationally equivalent to a Zariski closed subset of $\mathcal{C}$.*

We will prove this by explictly giving Φ and Ψ. The method used follows the main case of [5, §8, case (iv)].

So let f, g be quadratic functions in the three variables x, y, z, $\mathcal{C} = V(\{f,g\})$. Pick a random real solution $\hat{\mathbf{x}}$, as in §4, of the system $\{f = 0, g = 0\}$, such a solution exists by our assumption.

Now homogenize by $f_h = t^2 * f(\frac{x}{t}, \frac{y}{t}, \frac{z}{t})$ and similarly for g_h. Although we are thinking "projective curve" we really will be working in affine x, y, z, t space.

Now construct a random orthogonal 4×4 matrix A satisfying $A\hat{\mathbf{x}} = [0, 0, 0, 1]^T$. Using Algorithm 2 of §2 get homogeneous equations $\{f_1, g_1\}$ for the QSIC which is the image of $\mathcal{C}$ under the linear (projective) transformation $[x, y, z, t]^T \mapsto A[x, y, z, t]^T$. Since $[0, 0, 0, 1]$ is a solution of this system the coefficient of t^2 is 0 for both f_1, g_1. Collect the terms involving t and write

$$f_1 = tL + R \tag{2}$$
$$g_1 = tM + S \tag{3}$$

where L, M are linear in x, y, z and R, S are homogeneous quadrics in x, y, z. The linear polynomials L, M will be independent, and hence both non-zero, with probability 1; if this does not happen try a different random point and/or orthogonal matrix. Then $h_h = LS - RM$ is a hogeneneous cubic. Finally $h(x, y) = h_h(x, y, 1)$ is the desired plane cubic.

The forward map is

$$\Psi : \qquad [x, y, z] \mapsto A[x, y, z, 1]^T = [\check{x}, \check{y}, \check{z}, \check{t}\ ]^T \mapsto [\check{x}, \check{y}]/\check{z} \tag{4}$$

Whereas the backwards map makes use of (2) to recover t satisfying f_h, g_h from $[x, y, z]$:

$$\Phi : \quad [x, y] \mapsto A^{-1}\Big[x, y, 1, -\frac{R(x, y, 1)}{L(x, y, 1)}\Big]^T = [\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}\ ]^T \mapsto [\tilde{x}, \tilde{y}, \tilde{z}]/\tilde{t} \tag{5}$$

It is a straightforward check that, given our assumption, these maps do have the indicated domain and codomain with perhaps the exception of finitely many points.

To finsh the proof we need to calculate $\Psi \circ \Phi$. If we plug the last expression in the definition of Φ, $[\tilde{x}, \tilde{y}, \tilde{z}\ ]/\tilde{t}$, in for the first expression $[x, y, z]$ in the definition of Ψ then the first term is

$$\frac{1}{\tilde{t}}[\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}\ ]^T = \frac{1}{\tilde{t}}A^{-1}[x, y, 1, \tilde{t}]$$

Now this gets multiplied by A giving

$$\frac{1}{\tilde{t}}[x, y, 1, t] = \Big[\frac{x}{\tilde{t}}, \frac{y}{\tilde{t}}, \frac{1}{\tilde{t}}, \frac{t}{\tilde{t}}\ \Big]$$

by definition of Φ. But dropping the last two coordinates and dividing by the third leaves us with just $[x, y]$. Done.

We note that if our hypothesis of real affine QSIC does not hold the result is still true interpreted correctly in the complex projective situation. Even without this hypothesis the maps Φ, Ψ are still useful, however Φ may not be onto.

6 Parameterizing Plane quadratics and cubics

Since a line is immediately parameterizable, our problem is reduced to finding parameter patches for a plane cubic. We first check whether the plane cubic given by $h = 0$ is reducible.

Using Algorithm 2 above with just one point we can check for proper components and, since plane curves are solutions of single equations, components are associated with factors of the polynomial h. Any factors can be divided out making it easier to find any other factors. Thus we could still have to deal at this level with degree 1, 2 or 3 irreducible curves. Degree 1 is immediate. We will assume the irreducible curve is still called h.

Suppose $h = 0$ is the real equation of a plane curve of any degree and $\hat{\mathbf{p}}$ is a real non-singular point in $V(h)$. We will first use a fractional linear transformation to move $\hat{\mathbf{p}}$ to $[0, 1, 0]$ in $\mathbb{P}^2$ with the infinite line as tangent to the curve at this points. To this end we homogenize $h(x, y)$ to $h_h(x, y, z)$ and $\hat{\mathbf{p}}$ to $\hat{\mathbf{p}}_h$, in the latter case simply by adding a third coordinate 1. We define the normal vector at $\hat{\mathbf{p}}_h$ by

$$\mathbf{n} = \left[\frac{\partial h_h}{\partial x}, \frac{\partial h_h}{\partial y}, \frac{\partial h_h}{\partial z}\right]\Bigg|_{\hat{\mathbf{p}}}$$

It is well known, eg. [11, Chapter 4], that for a projective curve h_h at projective point $\hat{\mathbf{p}}_h$ then then $\mathbf{n} \cdot \hat{\mathbf{p}}_h = 0$.

Let $\mathbf{c} = \mathbf{n} \times \hat{\mathbf{p}}_h$ be the cross product and, simply for better numerical stability, let $\mathbf{n}, \hat{\mathbf{p}}_h, \mathbf{c}$ be normalized as $\bar{\mathbf{n}}, \overline{\hat{\mathbf{p}}_h}$ and $\bar{\mathbf{c}}$. Set B to be the 3×3 matrix

$$B = \begin{bmatrix} \bar{\mathbf{c}} \\ \overline{\hat{\mathbf{p}}_h} \\ \bar{\mathbf{n}} \end{bmatrix}$$

Then the fractional linear transformation given by B

$$\Theta : (x, y) \mapsto B[x, y, 1]^T = [\check{x}, \check{y}, \check{z}\ ]^T \mapsto [\check{x}, \check{y}\ ]/\check{z} \tag{6}$$

is our desired transformation. One should note that B is an orthogonal matrix.

Now if h is quadratic then by picking $\hat{\mathbf{p}}$ to be any point on $V(h)$ we arrive at a parabola $y = u(x)$ where $u(x) = ax^2 + bx + c$ is a quadratic in x since we have the unique infinite point $[0, 1, 0]$. In fact this is a real parabola even if h or $\hat{\mathbf{p}}$ are complex.

If h is an irreducible cubic curve, possibly singular, then we pick $\hat{\mathbf{p}}$ more carefully. In principle if h is singular we could pick $\hat{\mathbf{p}}$ to be the unique singular point. Now $\mathbf{n} = 0$ so we cannot use B above but any matrix B with $B.\hat{\mathbf{p}}_h = [0, 1, 0]$ would transform h into the form

$$y(dx + e) = ax^2 + bx + c \quad \text{ or } \quad y = \frac{ax^2 + bx + c}{dx + e}$$

giving a rational parameterization. In theory this should give satisfactory results but the hyperelliptic form below will behave much better numerically.

To get hyperelliptic form we take the Hessian curve H of h [11, §4.4] and find the intersections with h. As recently as 2000 when [16] was written this was computationally difficult, with modern numerical algebraic geometry, and even MATHEMATICA's numerical Gröbner basis, this is now routine. If h is irreducible and non-singular we are guarenteed at least one real point on $V(h_h) \cap V(H)$. If h is irreducible and singular but generated by the random procedure of Theorem 1 then it is still quite likely that there be such a non-singular point. This point will have an inflectional tangent. Using the fractional linear transformation from B above we move this point to $[0,1,0]$ with inflectional tangent the line at infinity, call the resulting homogeneous curve (use Algorithm 4) j_h. In theory the coefficients of y^3 and xy^2 of j_h are easily seen to be 0, it is known [5] that the coefficient of x^2y will also vanish because of the inflectional tangent. In numerical practice these coefficients will be very tiny so we can discard these terms. Specializing by setting $z = 1$ and dividing by the coefficient of y^2 gives

$$j = ax^3 + bx^2 + cx + dxy + ey + y^2 = y^2 + (dx + e)y + v(x)$$

For fixed x setting $j = 0$ the quadratic equation in y on the right has two solutions which add to $-dx - e$. In other words the line $y = -(\frac{d}{2}x + \frac{e}{2})$ lies on the midpoints of the two solutions of $j = 0$ for given x, possibly complex. The fractional linear transformation given by matrix

$$C = \begin{bmatrix} 1 & 0 & 0 \\ \frac{d}{2} & 1 & \frac{e}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

sends this line to the x-axis giving an equation of the form $y^2 = u(x)$ for some cubic polynomial $u(x)$. As in (6) the map from the h to the parameter curve is

$$\Theta : (x, y) \mapsto CB[x, y, 1]^T = [\check{x}, \check{y}, \check{z}\ ]^T \mapsto [\check{x}, \check{y}\ ]/\check{z}$$

A few more steps, using classical techniques actualized as fractional linear transformations, will put $y^2 = u(x)$ in the form $y^2 = x^3 + ax + b$ for suitable real a, b. But this is not necessary for our next step so we will not go into details here.

7 Parameterization and analysis of real QSIC

From the previous two sections we can parameterize each irreducible algebraic component of the QSIC in one of the following ways where u is a polynomial function of t of specified degree and the result is a rational function $\mathbb{R}^1 \to \mathbb{R}^3$ where each coordinate has the same denominator. The maximum degrees of numerator and denominator are given.

degree u	parameterization	max degrees
1	$\Phi(\Theta^{-1}(t, u(t)))$	degree 3 in t
2	$\Phi(\Theta^{-1}(t, u(t)))$	degree 6 in t
3	$\Phi(\Theta^{-1}(t, \pm\sqrt{u(t)}))$	degree 3 in t and $\sqrt{u}$

For degrees 1 and 2 we need only find the poles, i.e. zeros of the common denominator. The QSIC is of degree 4 so unless a component lies in the infinite plane of $\mathbb{P}^3$ it can interesect this plane in at most 4 points so at most 4 of these poles will be essential counting possible poles at the ends of the parameter lines. Thus the real projective line is divided into at most 4 intervals which would give up to 4 affine topological components. As the parameter goes to $\pm\infty$ and poles, even the inessential ones, the parameterization may become numerically inaccurate, thus it may be useful to use different choices and create an additional parameterization or two that would overlap to cover finite points of the QSIC that could be missed.

For degree 3 the situation is more complicated as for real QSIC the domain of the parameterizations is only part of the real line. Further because of the square root in the denominator it is harder to find the zeros. The first problem is easy as the domain is $\{t|u(t) \geq 0\}$ and it is only necessary to know the zeros of $u(t)$ to calculate this set. For the second collecting powers of $\sqrt{u(t)}$ one gets

$$v_0(t) + v_1(t)\sqrt{u(t)} + v_2(t)\sqrt{u(t)}^2 + v_3(t)\sqrt{u(t)}^3 \\ = v_0(t) + v_2(t)u(t) + \big(v_1(t) + v_3(t)u(t)\big)\sqrt{u(t)}$$

Setting this last expression equal to 0 gives

$$v_0(t) + v_2(t)u(t) = -\big(v_1(t) + v_3(t)u(t)\big)\sqrt{u(t)}$$

Squaring both sides gives

$$\big(v_0(t) + v_2(t)u(t)\big)^2 - \big(v_1(t) + v_3(t)u(t)\big)^2 u(t) = 0 \tag{7}$$

a single variable polynomial equation which is easily solved numerically. The zeros of the denominator are contained in the zero set of this equation, and note that this same zero set will apply to both parameterizations $y = \sqrt{u(t)}$ and $y = -\sqrt{u(t)}$ but denominators will have different zeros of (7). Combining this result with the calculation of the domain allows us to identify the affine topological components.

One posibility not yet considered is that two algebraic components may intersect. This will be a singular point of the QSIC Alternatively the image of a singular point in the QSIC likely appeared as a singular point in $V(h)$. One can identify the component curves containing this point then use the appropriate Θ to the parameter curves. If this singular point did not show up in $V(h)$ it is because of a line of the QSIC not in the image of Φ, which would show up in Step 1.

8 A fully worked out Example

In this section I do a complete example using the method outlined in the introduction. This is one of the more complicated examples.

Consider the QSIC given by $\mathcal{C} = V(\{f, g\})$ where

$$f = x^2 + z^2 - 2z - y^2, \quad g = 2x^2 - 2xy - 2z - 3xz \tag{8}$$

Step 1: Do numerical irreducible decomposition to identify complex algebraic components.

I use Bertini [2] on the system above which finds two algebraic components with non-singular witness points (rounded to 6 significant digits for display but given by 16 significant digits).

$$\hat{\mathbf{p}}_1 = (-.07985493 + .259423\imath, -.07985493 + .259423\imath, 0)$$
$$\hat{\mathbf{p}}_2 = (-.0978825 + .161370\imath, -.0164075 + .2438801\imath, .0215337 - .0120534\imath)$$

Bertini calculates the degree of the component containing $\hat{\mathbf{p}}_1$ to be 1, thus a line, and the degree of the other component to be 3.

The remaining steps are done with Mathematica with default precison, generally 17 digits, and linear algebra tolerance 10^{-12}.

Step 2: For this unbounded curve we can intersect with a random real plane to get the random real point $\hat{\mathbf{p}} = (1.133057, -0.5986927, 0.7268414)$ again rounded to 7 significant digits but used with 17 digits in the calculations.

Step 3: I next apply Theorem 1 to the system (8) to obtain the plane cubic

$$\begin{aligned} h = & -0.277749 + 0.471423x + 0.0905586x^2 - 0.00582857x^3 + 0.129608y \\ & - 0.0407206xy + 0.0475988x^2y + 0.362871y^2 - 0.0760948xy^2 + 0.0966969y^3 \end{aligned}$$

Here

$$A = \begin{bmatrix} 0.655457 & 0.407462 & 0.160413 & -0.61532 \\ 0.0261099 & -0.845622 & -0.00374475 & -0.53313 \\ 0.405925 & -0.0764678 & -0.898681 & 0.147482 \\ 0.636333 & -0.33623 & 0.408199 & 0.561607 \end{bmatrix}$$

$$L = -0.511267x + 0.252642y + 0.229098z$$

$$R = -0.165283x^2 - 0.174560xy + 0.291000y^2 + 0.506095xz + 0.400222yz - 0.252014z^2$$

Thus from (4) the map $\Psi : \mathcal{C} \to V(h)$ is given by

$$\Psi(x, y, z) = \left(\frac{\alpha_1(x, y, z)}{\gamma(x, y, z)}, \frac{\alpha_2(x, y, z)}{\gamma(x, y, z)} \right)$$

where

$$\begin{aligned} \alpha_1(x, y, z) = & -0.61532 + 0.655457x + 0.407462y + 0.160413z \\ \alpha_2(x, y, z) = & -0.53313 + 0.0261099x - 0.845622y - 0.00374475z \\ \gamma(x, y, z) = & 0.147482 + 0.405925x - 0.0764678y - 0.898681z \end{aligned}$$

And from (5) the map $\Phi : V(h) \to \mathcal{C}$ is given by

$$\Phi(x, y) = \left(\frac{\beta_1(x, y)}{\delta(x, y)}, \frac{\beta_1(x, y)}{\delta(x, y)}, \frac{\beta_1(x, y)}{\delta(x, y)} \right)$$

where

$$\begin{aligned} \beta_1(x, y) = & 0.253361 - 0.379418x - 0.229939x^2 - 0.146139y + 0.263325xy - 0.178576y^2 \\ \beta_2(x, y) = & -0.102253 + 0.302609x - 0.263895x^2 - 0.0784822y + 0.476589xy - 0.115797y^2 \\ \beta_3(x, y) = & -0.103014 + 0.289628x - 0.0145455x^2 - 0.391273y + 0.113697xy - 0.119732y^2 \\ \delta(x, y) = & 0.175321 - 0.500598x + 0.407417x^2 - 0.309646y + 0.21515xy - 0.298119y^2 \end{aligned}$$

Step 4: We pick random linear form $k = 0.647639x + 0.134229y$ then the Jacobian Determinant is the ellipse

$$\mathcal{J} = -0.0206603 + 0.0506834x - 0.0331739x^2 - 0.475485y + 0.111342xy - 0.198088y^2$$

which intersects h in a singular point $(-4.17438, -3.61523)$ and points $q_1 = (0.537437, -0.00725009), q_2 = (-6.75208, -2.73808)$. Applying Algorithm 1 to q_1 decomposes $h = h_1 h_2$ into the two curves

$$\begin{aligned} h_1 =& 0.407209 - 0.670328x - 0.167057x^2 - 0.326707y + 0.267999xy - 0.422342y^2 \\ h_2 =& -0.682079 + 0.0348897x - 0.228954y \end{aligned}$$

Applying Ψ to the two points $\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2$ we find that $\hat{\mathbf{p}}_1$ maps to a point on $V(h_2)$ while $\hat{\mathbf{p}}_2$ maps to a point on h_1. Thus Ψ respects the decompositions of $\mathcal{C}$ and $V(h)$ so we can conclude that Ψ is a numerical birational equivalence of curves $\mathcal{C}, V(h)$ with inverse Φ.

Step 5: Using the method of §6 we get a fractional linear transformation Θ with matrix

$$B = \begin{bmatrix} 0.545499 & 0.668587 & 0.505394 \\ -0.832815 & 0.364724 & 0.416408 \\ -0.0940751 & 0.64805 & -0.755766 \end{bmatrix}$$

which takes $V(h_1)$ to the parabola

$$y = 0.145433 - 0.209493x - 0.322363x^2 = u(x)$$

Then our parameterization of the second component of $\mathcal{C}$ is

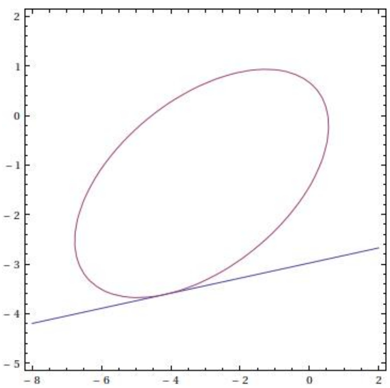
$$\Phi\Theta^{-1}((t, u(t)) = \left(\frac{\xi_1(t)}{\rho(t)}, \frac{\xi_2(t)}{\rho(t)}, \frac{\xi_3(t)}{\rho(t)}\right)$$

Where

$$\begin{aligned} \xi_1(t) &= -0.00123344 + 0.116394t + 0.0455282t^2 - 0.0600163t^3 - 0.0114211t^4 \\ \xi_2(t) &= -0.106938 + 0.0555114t + 0.165079t^2 - 0.0241627t^3 - 0.0496511t^4 \\ \xi_3(t) &= 0.107571 - 0.116773t - 0.0431373t^2 + 0.0859286t^3 - 0.0247614t^4 \\ \rho(t) &= 0.000638119 - 0.06223t + 0.165178t^2 + 0.227341t^3 + 0.0347652t^4 \end{aligned}$$

The first component of $\mathcal{C}$ is easily seen to be the line $y = x, z = 0$. The two components meet at the origin of $\mathbb{R}^3$ which is Φ of the singular point of $V(h)$ where the two components of $V(h)$ meet tangentially.

Step 6: Solving $\rho(t) = 0$ we get $\{-5.64066, -1.17225, 0.010554, 0.26302\}$, the root $t = -1.17225$ is not an essential pole but the others are. To graph the second component we need only plot the line and $\Phi\Theta^{-1}(t, u(t))$ on the real intervals $[-100, -5.67], [-5.6, .01], [.011, .26], [.27, 5]$. There is a tiny gap near the singular point $(0, 0, 0)$ but it is not very noticible on the graph (Figure 2).



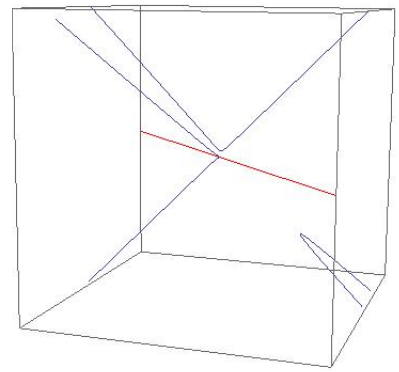


Figure 2: $V(h)$ (left) and the QSIC inside $-40 \leq x, y, z \leq 40$.

I must make one further comment about this example. The algebraic component which is not a line is of degree 3 but not planar. Thus it cannot be a complete intersection curve in $\mathbb{C}^3$. In fact the surface given by

$$\gamma = 0.40756x - 0.649881x^2 - 0.40756y - 0.391031xy - 0.181769y^2 \\ - 0.143229z - 0.0770954xz + 0.181769z^2$$

is linearly independent of f, g but also contains this degree 3 component. Thus the decomposition of Step 1 is not helpful in finding a parameterization for this component. The decompositions of Step 1 and Step 4 serve different purposes, that of Step 1 identifies components to see if Ψ is onto while Step 4 decomposes the curve into useful complete intersection components. In this example both of these are necessary.

In [15] the classification of QSIC gives 23 of the 35 total types which are reducible with only planar components. In these 23 cases one could go directly from Step 1 to Step 5. In some of the other cases the QSIC is irreducible in which case the information from Step 1 allows us to skip Step 4. But for a complete description of all QSIC we need this 6 step method.

9 More Examples

Examples have been calculated with a default of approximately 17 digits in Mathematica 8 with a numerical linear algebra tolerance of 10^{-12}. Generally the calculations are accurate to about 11 digits but for display only about 6 digits are shown here. It should be noted that because a number of random choices are made in this approach that the details of these examples can not be replicated without knowing the choices. But the point set and properties of the QSIC obtained will be the same with different random choices.

Example 1: Generically a QSIC will be a genus 1 space curve. A typical example is $V(\langle f, g \rangle)$

$$f = x^2 + y^2 + z^2 - 16, \quad g = 57 - 12x + 4x^2 + y^2 - 64z + 16z^2 \tag{9}$$

An application of Theorem 4 gives

$$
\begin{aligned}
h = 0.0442427 &+ 0.140313x + 0.116615x^2 + 0.0294054x^3 \\
&- 0.217169y - 0.402909xy - 0.172791x^2y + 0.344722y^2 \\
&+ 0.330695xy^2 - 0.208248y^3
\end{aligned}
$$

Checking h there are no singularities and three real inflectional tangents two of which, pictured below in the top lefthand plot in Figure 1, then give, using the method of §6, the two curves

$$
\begin{aligned}
y^2 &= 38.6067 + 16.0287x + 6.07442x^2 + 0.716794x^3 \\
y^2 &= -2.33478 + 1.87741x - 0.375185x^2 + 0.0245423x^3
\end{aligned}
$$

both of which look like the upper right picture of Figure 3 but with very different scales. Note these curves are both birationally equivalent to the QSIC (9) and so are birationally equivalent to each other. Each of these curves gives two patches (positive and negative y) the the four patches cover QSIC (9)

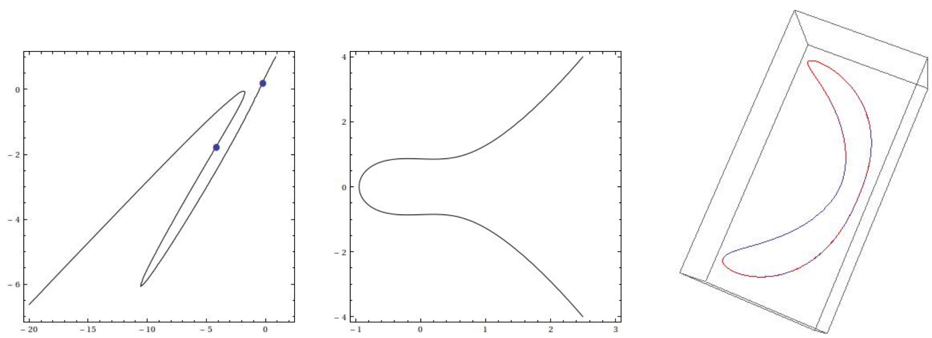


Figure 3: Example 1, bottom curve is QSIC

Example 2: A more complicated version of the above is the curve $V(\langle f, g \rangle)$

$$
f = x^2 - y^2 + z^2 - 1 \quad g = x^2 - z^2 - 4 \tag{10}
$$

The plane cubic obtained is

$$
\begin{aligned}
h = -0.0947177 &- 0.263213x + 0.267739x^2 - 0.0855912x^3 \\
&- 0.308632y + 0.0559627xy - 0.194258x^2y \\
&- 0.122697y^2 + 0.387761xy^2 - 0.0316547y^3
\end{aligned}
$$

which is transformed into

$$
\begin{aligned}
y^2 &= -0.0029057 - 0.494465x + 0.252846x^2 + 0.45218x^3 \\
&= u(x)
\end{aligned}
$$

The parameter domain is then $\{-1.359 \leq t \leq 0.00586\} \cup \{0.8065 \leq t\}$ and the possible zeros from (7) of the parameter functions $y = \pm\sqrt{u(t)}$ are $t = -1.2287, -.05820, 0.8936, 1.347, 20.925$. The values $t = -1.2287, .8936$ are poles of $y = \Phi\Theta^{-1}(t, \sqrt{u(t)})$ while $t =$

$-.05197, 20.925$ are poles of $y = \Phi\Theta^{-1}(t, -\sqrt{u(t)})$ but $t = 1.3465$ turns out not to be an essential pole of either. Thus our parameter curve breaks into 4 intervals as shown, dots are poles, on the left in Figure 3 below. The images under $y = \Phi\Theta^{-1}(t, \pm\sqrt{u(t)})$of the intervals are the 4 affine topological components of the QSIC as shown on the right of Figure 4 after projection by $(x, y, z) \mapsto (x + .5z, y)$. Note we used Algorithm 4 to find the equation of the projection and plotted with a contour plot.

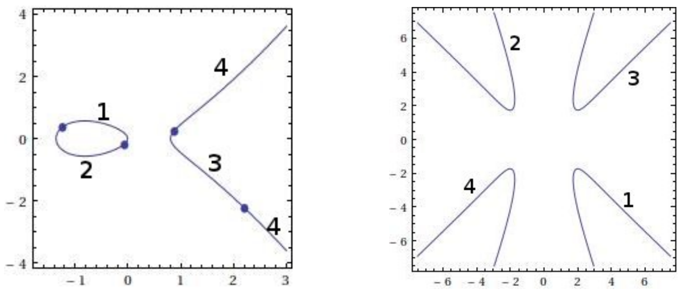


Figure 4: top parameter curve, bottom projection of QSIC in Example 2

Example 3: Consider curve number 6 in [15, Table 1] $V(\langle f, g\rangle)$ where

$$f = -x^2 - z^2 + 2y, \quad g = -3x^2 + y^2 - z^2 \tag{11}$$

This curve has a bounded dimension 1 real component and an isolated real point. Theorem 4 gives the plane cubic

$$\begin{aligned} h = 0.00491366 - 0.0192757x + 0.304379x^2 + 0.13571x^3 \\ + 0.0803394y + 0.449203xy + 0.11988x^2y \\ + 0.661842y^2 - 0.0941065xy^2 + 0.0385216y^3 \end{aligned}$$

This h also has an isolated real point which is a singular point. Letting h_h be the homgenation of h MATHEMATICA'S `NSolve`, in a rare miss, does not see the common zero $\hat{\mathbf{x}} = [0.103102, -0.0989506, 1]$ of $\{\frac{\partial h_h}{\partial x}, \frac{\partial h_h}{\partial y}, h_h\}$ and also fails to identify $\hat{\mathbf{x}}$ as a multiple real zero of the intersection of h_h and its Hessian, seeing instead two close complex zeros. Thus it is not unexpected that one may mistake h for a non-singular cubic. Since there are real inflectional points the method of §4 gives a birational equvalence of h with $y^2 = 0.498 + 2.55083x + 2.54258x^2 - 1.94738x^3$ which is a singular curve with isolated real singular point at $(-0.356012, 0)$. This curve maps birationally onto the QSIC (11) including the isolated point and we get, a good parameter patch.

With the method of §4 the isolated point of both the original QSIC and h are easily found, see the example in §4.

So h must be a singular curve but from the equation alone it is difficult in some cases to identify whether a numerical curve is singular. Thus this suggested method which does not treat singular and non-singular curves diffently is preferable to a method which treats them as separate cases. The singular isolated point is more readily identified in the final $y^2 = u(x)$ form.

The reader should notice the difference here between working numerically and exactly. In an exact calculation h could be a curve of the form $y - v(x)$ in which case it would make no sense transforming it to the cusp $y^2 - u(x)$ to get a parameterization. In our numerical case the chance of h being of the form $y - v(x)$ is virtually nil so it is not worth even considering that possibility.

Example 4: The easiest example of a QSIC, the intersection of two spheres, is one not covered by Theorem 4 as stated, since it is not an affine QSIC. for instance

$$f = x^2 + y^2 + z^2 - 4, \qquad g = -2 - 2x + x^2 + y^2 + z^2 \tag{12}$$

Note $f - g = 2x - 2$ so the affine solution is $\{x = 1, y^2 + z^2 = 3\}$. However applying the technique of Theorem 4 gave the cubic

$$\begin{aligned} h = 0.15022 + 0.10600x + 0.0724682x^2 - 0.00712392x^3 \\ + 0.44953y + 0.31137xy + 0.156986x^2y \\ + 0.403442y^2 + 0.165618xy^2 + 0.206231y^3 \end{aligned}$$

The curve $V(h)$ is reducible with components $V(h_1), V(h_2)$

$$\begin{aligned} h_1 &= 0.448791 - 0.038654x + 0.8928y \\ h_2 &= 0.334722 + 0.265018x + 0.1843x^2 + 0.335768y \\ &\quad +0.195505xy + 0.230994y^2 \end{aligned}$$

where Φ applied to the line $h_1 = 0$ parameterizes the real affine solution circle but the quadratic h_2 has no real solutions. However Φ does not take $V(h_2)$ to $V(\{f, g\})$. Instead if we work projectively then the projective closure of $V(h_2)$ goes to the complex ideal curve in $\mathbb{P}^3$ given by $\{t = 0, x^3 + y^3 + z^3 = 0\}$ which is in the projective closure of (12).

Example 5: Another simple example is

$$f = x^2 + y^2 - 1 \quad g = z - xy \tag{13}$$

By inspection I could easily parameterize the affine real part using the unit circle by $(x, y) \mapsto (x, y, xy)$. But $V(f, g)$ is not a plane curve and is irreducible of degree 4 as a complex projective variety. In fact this variety has an isolated real point at infinity.

Using the method of this paper I can parameterize this curve by the singular plane cubic

$$y^2 = -0.0573176 - 0.67692x + 0.637452x^2 - 0.144568x^3$$

with isolated real point $(2.24404, 0.)$.

Example 6: The system $\{x^2 - y^2, z^2 - 1\}$ [15, curve 28] consists of 4 real affine lines and does satisfy the hypotheses of Theorem 4 but it is impossible for Φ to be a birational equivalence because a cubic cannot have 4 components.

10 Conclusion

This paper shows that working numerically instead of exactly greatly simplifies the problem and yet we are still able, in our experiments, to distinguish the different types of QSIC, even when there are singular points involved. It is possible, of course, that very sensitive examples may be found where our method may need higher precision or arithmetic or may fail.

Using the entire algorithm, especially Step 5, we can specialize Theorem 1 as follows:

Theorem 2 *Suppose $\mathcal{X}$ is an irreducibe real affine QSIC, possibly singular. Then $\mathcal{X}$ is numerically birationally equivalent to a plane cubic $\mathcal{Y}$ in hyperelliptic form $y^2 = x^3 + ax + b$ for real numbers a, b.*

Numerically birationally equivalent means that there are birational maps from $\mathcal{X}$ to $\mathcal{Y}$ and back where the coefficients are floating point reals rather than rational numbers. It is still a question in my mind as to what algebraic -geometric data is preserved by such an equivalence. As a former pure mathematican I am tempted to answer "very little". Yet the more I work with these numerical methods the more I am impressed with the power of them.

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