How many panels in a traditional Soccer Ball?

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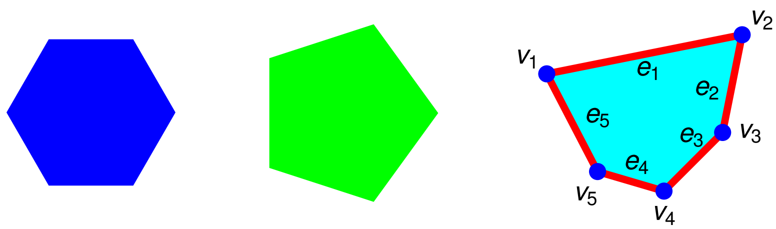
The traditional soccer ball was made from separate panels consisting of black hexagons and white pentagons. How many of each to use intuitively should depend on the size of the ball and size of individual panels. But this is wrong, the numbers of panels is independent of size. In fact we can count the number from just knowing the configuration of pentagons and hexagons in the picture above. This depends on a theorem due to Leonhard Euler (1707-1783) known as *Euler's Polyhedron Formula.*

1. Euler' s Polyhedron Formula for Spherical Polyhedra

The traditional soccer ball is an example of a spherical polyhedron. A general spherical polyhedron is a solid finite figure without holes, for example a donut (torus) does not qualify, where the surface consists of faces which are approximately convex plane polygons, not necessarily all the same or regular. Each face has sides, we call them edges, which are approximately straight line segments and the edges meet in vertices. The faces are convex polygons, in particular they do not have holes, in fact any two points inside the face can be connected by a line lying entirely inside the face. Spherical means the entire surface could be approximated on a sphere.

Examples of polygons are hexagons, pentagons and other sorts of shapes.

Out[ ]=



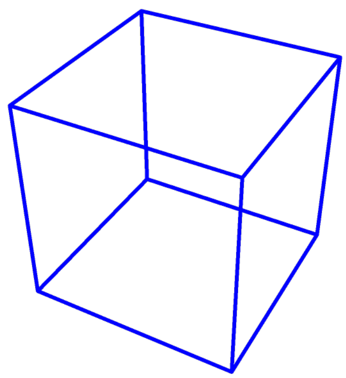
The last polygon shows the vertices and edges, there are 5 of each. We will use V for the number of vertices, E for the number of Edges, and F for the number of faces.

Euler' s Spherical polyhedron Theorem:

$$F - E + V = 2$$

We will discuss why this is true later. For now a simple example is a cube.

Out[∘]=



We see there are 8 vertices, 12 edges and 6 faces.

In[∘]:= **8 – 12 + 6**

Out[∘]= 2

2. Number of Pentagons and Hexagons on traditional Soccer Ball

Let p be the number of pentagons (black), h the number of hexagons {white). Since each face of this spherical polyhedron is either a pentagon or hexagon we have $F = p + h$. Notice that each edge is an edge of two polygons, pentagon or hexagon. The total number of such edges is $5\,p + 6\,h$ so that counts the actual number of edges twice so the number of edges is $\frac{6\,h+5\,p}{2}$. Finally there are two ways to count the vertices. We notice that each vertex is a pentagon vertex, so $V = 5\,p$. On the other hand we notice each vertex is a vertex of 2 hexagons so the number is $\frac{6\,h}{2}$ = 3 h.

If we use the first formula for the number of vertices Euler' s theorem above gives

$$F - E + V = (p+h) - (6\,h + 5\,p)/2 + 5\,p = 2$$

But we also have the equation $5\,p = 3\,h$

$$h = 5\,p/3$$

which we can plug into our first equation to get left hand side

In[ ]:= **Simplify[(p + (5 p / 3)) – (6 × (5 p / 3) + 5 p) / 2 + 5 p]**

Out[ ]= $\frac{p}{6}$

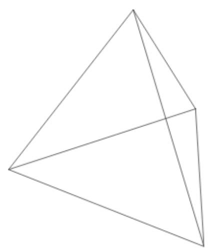
So our first equation is simply $\frac{p}{6} = 2$ or $p = 12$. But then $h = 5\,p/3 = 5 \times 12/3 = 20$

So we always have exactly 12 pentagons and 20 hexagons, or 32 panels.

3. Why Euler' s Theorem works

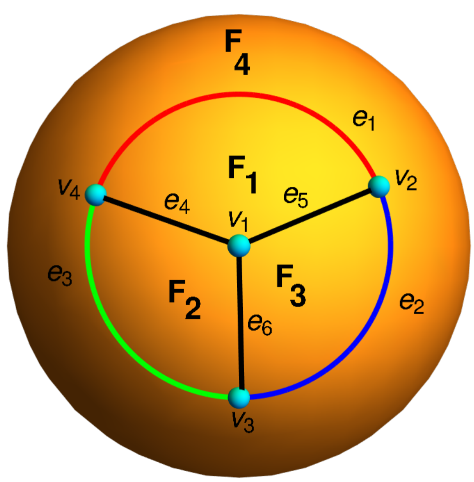
In this paper *Euler' s Theorem* refers to the theorem in Section 1. An important hypothesis is that the polyhedron is spherical, that is a representative can be drawn on the sphere. Each point on the sphere must be a vertex, in an edge or in a face.

The simplest spherical polyhedron is the tetrahedron with 4 faces, 6 edges and 4 vertices. $4 - 6 + 4 = 2$



From a spherical point of view this can be represented by

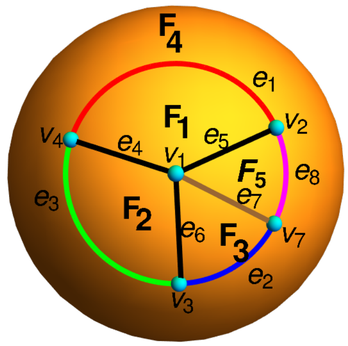
Out[ ]=



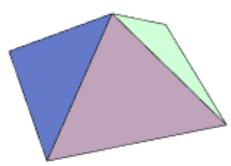
where the bottom face, base, consists of the entire outside of the circle.

We can modify this . Lets add another vertex v_7 on the base and an edge from the apex v_1 to this new vertex v_7 so it will be a pyramid with rectangular base. But this splits the former edge e_2 into two parts, one will remain e_2 but the new will now be e_8. This produces a new face F_5.

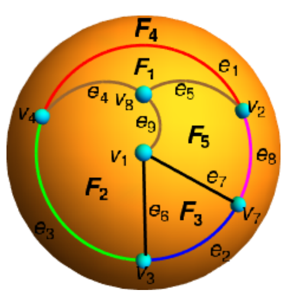
Out[∘]=



But we have added a vertex and a face but two new edges. So Euler's formula becomes $5 - 8 + 5 = 2$ which gives a pyramid with rectangular base.



Next we add a new vertex v_8 and a new edge e_9. We move the inside of edges e_4 and e_5 from v_1 to our new v_8. This change does not create any new faces. Now we have a prism with 5 faces, 9 edges and 6 vertices, $5 - 9 + 6 = 2$.



Graphics by Mathematica

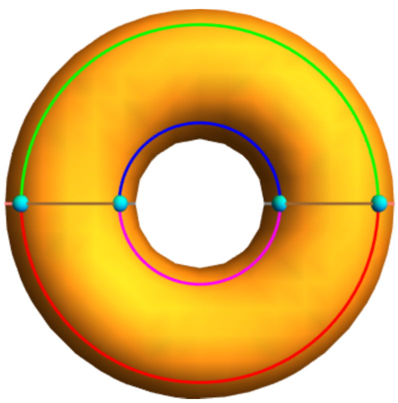
Pictures from Wolfram Mathworld

One can continue this way adding or deleting vertices and edges on the sphere as long as there are no hanging or loose edges. That means there can not be an edge which has a vertex not attached to another edge or an edge which does not share a vertex with our previous polygon. New faces may be generated. But as in the examples above such constructions will always leave us with the equation F - E + V = 2.

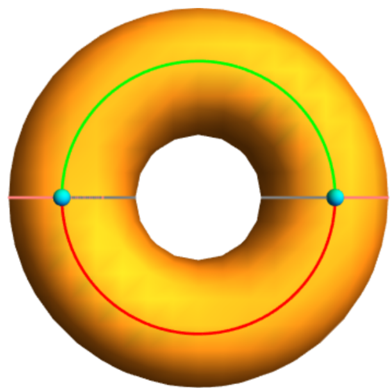
4. When Euler's Theorem does not work.

An important hypotheses in Euler's theorem is that the polyhedron is spherical, that is we can create a model on the sphere. In fact mathematicians see Euler’s theorem as saying something about the geometry, more precisely, the topology of the sphere. If one attempts to draw a similar decomposition of a different sort of surface into simple convex regions then instead of the 2 on the right hand side we may get another number, this number helps to describe different surfaces and is called the *Euler Characteristic.*

For example suppose we try this on a Torus .



Top



Bottom

Here we have 6 faces, we 2 clearly on the top and 4 on the bottom. We see 6 edges on the top and 6 on the bottom counting the 2 pink edges at the side which go from the top to the bottom for a total of 12. And finally there are 6 vertices. So

$$F - E + V = 6 - 12 + 6 = 0$$

Note that if we add new vertices and edges as the examples above, again because of the creation of new faces, the equation will still balance. Hence the zero on the right hand side is a characteristic of the surface shape “torus”. That is the torus has Euler Characteristic 0.