Numerical Calculation of H-Bases for Positive Dimensional Varieties

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H-Basis

- **Möller and Sauer**

  At a very early time, when even the notion of ideals was not commonly accepted, Macaulay introduced H-bases*. These special bases of polynomial ideals are also helpful in various branches of numerical analysis.*

- **Macaulay**

  The distinctive property of an H-basis $(F_1, F_2, \ldots, F_k)$ *of* M *is than any member* F *of* M *can be put in the form* $A_1F_1 + A_2F_2 + \cdots + A_kF_k$ *where* A_iF_i $(i = 1, 2, \ldots, k)$ *is not of greater degree than* F*. Every module [ideal] has an H-basis, which may necessarily consist of more members than would suffice for a basis in general*

Macaulay and Sylvester Arrays

Let $f = x - y,\ g = z + x^2 - y^2$ in $\mathbb{C}[x, y, z]$, $\quad F = [f, g]$

Macaulay Array of F **of order 2 at** $(0, 0, 0)$

	1	x	y	z	x^2	xy	xz	y^2	yz	z^2
f	0	1	-1	0	0	0	0	0	0	0
g	0	0	0	1	1	0	0	-1	0	0
xf	0	0	0	1	-1	0	0	0	0	0
yf	0	0	0	0	0	1	0	-1	0	0
zf	0	0	0	0	0	0	1	0	-1	0
xg	0	0	0	0	0	0	1	0	0	0
yg	0	0	0	0	0	0	0	0	1	0
zg	0	0	0	0	0	0	0	0	0	1

Note that rows xg, yg, zg are truncated.

Macaulay and Sylvester Arrays (continued)

Sylvester Array of F of order 2, $\mathbf{S}(F, 2)$

	1	x	y	z	x^2	xy	xz	y^2	yz	z^2
f	0	1	-1	0	0	0	0	0	0	0
g	0	0	0	1	1	0	0	-1	0	0
xf	0	0	0	1	-1	0	0	0	0	0
yf	0	0	0	0	0	1	0	-1	0	0
zf	0	0	0	0	0	0	1	0	-1	0

Sylvester Array of ideal $\langle f, g \rangle$ of order 2, $\mathbf{S}(\mathcal{I}, 2)$

	1	x	y	z	x^2	xy	xz	y^2	yz	z^2
$\mathbf{x} - \mathbf{y}$	0	1	-1	0	0	0	0	0	0	0
$\mathbf{z}$	0	0	0	1	0	0	0	0	0	0
$x(x-y)$	0	0	0	0	1	-1	0	0	0	0
$y(x-y)$	0	0	0	0	0	1	0	1	0	0
xz	0	0	0	0	0	0	1	0	0	0
yz	0	0	0	0	0	0	0	0	1	0
z^2	0	0	0	0	0	0	0	0	0	1

Global and Local dual functionals

Let $\mathcal{I}$ be an ideal of $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_s]$, the local ring at $\hat{\mathbf{x}} = 0$ is $\mathbb{C}[[x_1, \dots, x_s]]/\,\mathbb{C}\,[[x_1, \dots, x_s]]\mathcal{I}$. For $\mathbf{j} = [j_1, \dots, j_s], \mathbf{x}^{\mathbf{j}} = x_1^{j_1} \dots x_s^{j_s}$.

A *Global dual functional* is a $\mathbb{C}$-linear map

$$\mathbb{C}[\mathbf{x}]/\mathcal{I} \longrightarrow \mathbb{C}$$

A typical such functional is given by an **infinite** sum

$$\sum_{\mathbf{j}} \alpha_{\mathbf{j}} \mathbf{X}^{\mathbf{j}} \text{ where } \mathbf{X}^{\mathbf{j}}(\mathbf{x}^{\mathbf{k}}) = \begin{cases} 1 & \text{if} \quad \mathbf{j} = \mathbf{k}, \\ 0 & \text{if} \quad \mathbf{j} \neq \mathbf{k}. \end{cases}$$

A *Local dual functional* is a $\mathbb{C}$-linear map

$$\mathbb{C}[[\mathbf{x}]]/\mathbb{C}[[\mathbf{x}]]\mathcal{I}\Big|_{\hat{\mathbf{x}}} \longrightarrow \mathbb{C}$$

A typical such functional is given by a **finite** sum

$$\sum_{|\mathbf{j}|<n} \beta_{\mathbf{k}} \partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}], \text{ where } \partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}] \equiv \frac{1}{j_1! \cdots j_s!} \left. \frac{\partial^{j_1+\cdots+j_s}}{\partial x_1^{j_1} \cdots \partial x_s^{j_s}} \right|_{\hat{\mathbf{x}}}$$

Global and Local dual functionals spaces as arrays

Local duals can be put in Sylvester type arrays, global in Macaulay type. We view the dual functionals as columns.
Consider the ideal $\langle f \rangle \subseteq \mathbb{C}[x,y]$ given by
$f = x + 2y + x^2 + 3xy + y^2$. The local duals are at point $(0,0)$, indices on right.

Local duals order 2

1	0	0	∂_1
0	-2	1	∂_x
0	1	0	∂_y
0	0	4	∂_{x^2}
0	0	-2	∂_{xy}
0	0	1	∂_{y^2}

Global duals order 2

1	0	0	0	0	1
0	-2	1	1	0	X
0	1	0	0	0	Y
0	0	4	-4	-3	X^2
0	0	-2	1	1	XY
0	0	1	0	0	Y^2

Note the last two columns of the global duals are truncated.

Local and Global Duality

The notion of global dual functional $\mathbb{C}[\mathbf{x}]/\mathcal{I} \to \mathbb{C}$ implies that dual functionals kill the ideal $\mathcal{I}$ (and similarly for local duals).
From this viewpoint say matrices A, B are *dual* if $AB = 0$ with rowspace A the left nullspace of B and column space B the nullspace of A.

The Sylvester array of local duals is dual to the Macaulay matrix while the the Macaulay array of Global duals is dual to the Sylvester array of the ideal.

Local to Global

For $\mathbf{i} = [i_1, \ldots, i_s], \mathbf{j} = [j_1, \ldots, j_s]$, $\mathbf{i} \geq \mathbf{j}$ means $i_\alpha \geq j_\alpha$ for all $1 \leq \alpha \leq s$. Then as functionals on $\mathbb{C}[\mathbf{x}]$

$$\partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}] = \sum_{\mathbf{i} \geq \mathbf{j}} \binom{i_1}{j_1} \hat{x}_1^{i_1 - j_1} \cdots \binom{i_s}{j_s} \hat{x}_s^{i_s - j_s} \mathbf{X}^{\mathbf{i}}$$

where $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$. The left hand side is a local functional and the right a global functional. From a matrix point of view we have for fixed n

$$\begin{bmatrix} \text{Macaulay Matrix} \\ \text{of order } n \\ \text{global duals from } \hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \text{Change of Center} \\ \text{matrix} \\ \text{of order } n \end{bmatrix} \begin{bmatrix} \text{Sylvester Matrix} \\ \text{of order } n \\ \text{local duals at } \hat{\mathbf{x}} \end{bmatrix}$$

Local to Global

Change of Center Matrix

For example if $s = 2$ and $\hat{\mathbf{x}} = (1, 2)$ then the *Change of center matrix* is

$$\gamma_{\hat{\mathbf{x}}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 4 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 4 & 4 & 4 & 0 & 4 & 1 & 0 & 0 & 1 & 0 \\ 8 & 0 & 12 & 0 & 0 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Local to Global, Main Theorem

Given an ideal $\mathcal{I}$ of $\mathbb{C}[x_1, \ldots, x_s]$, $n > 0$ and points $\hat{\mathbf{p}}_i,\ i = 1, \ldots, k$ of $V(\mathcal{I})$ concatenate the Macaulay matrices of order n global duals. Write $\mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ for this matrix.

Main Theorem, matrix form: *For given $n > 0$ there exist finitely many points, $\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k$, of $V(\mathcal{I})$ so that the Sylvester matrix of the ideal $\mathcal{I}$ of order n is the left nullspace of $\mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$.*

Corollary *An H-Basis for $\mathcal{I}$ can be obtained from finitely many local duals at finitely many points of $V(\mathcal{I})$.*

It remains an open question as to how many and what points are needed. It is clear that it is necessary to have at least one point from each component of $V(\mathcal{I})$. In principle, for large n, that may be enough. In practice more points may be needed and the number may be dependent on implementation issues as well as algebraic-geometric factors.

The global Hilbert function

The global Hilbert function is

$$\mathrm{GHF}(n) = \binom{n+s}{s} - \mathrm{rank}\ \mathbf{S}(\mathcal{I}, n),\ \ n > 0$$

But if $\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k$ satisfy the Main Theorem then

$$\mathrm{GHF}(n) = \mathrm{rank}\ \mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\}),\ \ n > 0$$

For large n the values of the Hilbert function agree with a integer valued polynomial known as the Hilbert Polynomial. This can often be calculated independently. In particular the leading term $c_d t^d$ of this polynomial can often be calculated by standard Numerical Algebraic geometry software. In general one wants to pick points and tolerances minimizing the global Hilbert function retaining the correct leading term of the Hilbert Polynomial.

Algorithms

The algorithms for finding $\mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ [see also Mourrain, Li and Zhi, Zeng] and $\mathbf{S}(\mathcal{I}, n)$ are straight forward.

Two algorithms have been used for extracting H-bases.
MBasis1

- Calculate $\mathcal{D}_N(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ for N large enough that $\mathbf{S}(\mathcal{I}, N)$ contains an H-Basis. For $n \leq N$ $\mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ is truncation of $\mathcal{D}_N(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ and $\mathbf{S}(\mathcal{I}, n) = \text{left nullspace } \mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$.
- For $n = 1, 2, \ldots$ calculate $\mathbf{S}(\mathcal{I}, n)$ until $\mathbf{S}(\mathcal{I}, n_0)$ is non-empty. Interpret entries of $\mathbf{S}(\mathcal{I}, n_0)$ as polynomials and set $\mathcal{B}_{n_0}$ to be this list of polynomials.
- For $n_0 < n \leq N$ note $\mathbf{S}(\mathcal{B}_{n-1}, n) \subseteq \mathbf{S}(\mathcal{I}, n)$. If this inequality is an equality set $\mathcal{B}_n = \mathcal{B}_{n-1}$. Otherwise there are rows of $\mathbf{S}(\mathcal{I}, n)$ independent of $\mathbf{S}(\mathcal{B}_{n-1}, n)$ and add corresponding polynomials to $\mathcal{B}_{n-1}$ to obtain $\mathcal{B}_n$.
- **Output:** $\mathcal{B}_N$

Algorithms

MBasis2

Example: $\mathcal{I} = \langle 1 + x + y + xy^2, 1 - y^2 \rangle$

Calculate $\mathbf{S}(\mathcal{I}, 3)$, put this in reverse RREF form:
(monomial order $1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3$)

$$\left[\begin{array}{cccccccccc}
\mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

The first row is $\mathbf{S}(\mathcal{I}, 1)$, the first 4 rows form $\mathbf{S}(\mathcal{I}, 2)$ and the entire matrix is $\mathbf{S}(\mathcal{I}, 3)$. As in algorithm **MBasis1** the minimal H-basis is $1 + 2x + y,\ x + x^2$.

In both algorithms numerical issues have been ignored, see paper.

Applications, irreducible decomposition (factoring)

Using Mathematica 6

Function to factor or reducible variety $V(\boldsymbol{f})$ to decompose.
f$=2+3x^2-5x^4+x^6-3y^2-16x^2y^2+8x^4y^2-5y^4+16x^2y^4+7y^6+9xz-4x^3z-2x^5z-22xy^2z+11xy^4z+7x^2z^2-4x^4z^2-11x^2y^2z^2+x^3z^3$

Pick x, y randomly, solve for z on surface $V(\boldsymbol{f})$
P = NSolve[{f, {x,y}-RandomComplex[{-1-I,1+I},2]}]

$\boldsymbol{p1} = \{-0.2477+0.897805i, -0.59133-0.126784i, 1.03993+5.73923i\}$
$\boldsymbol{p2} = \{-0.2477+0.897805i, -0.59133-0.126784i, 0.232135-0.184349i\}$
$\boldsymbol{p3} = \{-0.2477+0.897805i, -0.59133-0.126784i, 0.395344+1.01247i\}$

Find global duals up to order 7 corresponding to local duals at **p1**.
corresponding to local duals at **p1**.
Timing[G1 = GDiff[{f}, {$p1$}, 7, X, eps];]
$\{1.75, \mathrm{Null}\}$
Calculate Hilbert function
Timing[AFHF[G1,7, 3, eps]]
$\{0.08, \{1, 4, \mathbf{9}, 16, 24, 30, 34, 36\}\}$

Note Hilbert function is deficient by 1 in order 2.

Irreducible decomposition (continued)

Find quadratic of degree two in ideal of component containing $\hat{\mathbf{p}}_1$.

Timing[g1 = Round[Chop[Re[SBasis[G1, 2, 3, eps]], 1.*^-9], 1.*^-9][[1]].X2]

$\{0., -0.447561736 + 0.364721029x^2 + 0.812282765y^2 - 0.082840707xz\}$

Similarly we get factors (rounded to nine digits):

$\mathbf{g1} = -0.447561736 + 0.364721029x^2 + 0.812282765y^2 - 0.082840707xz$

$\mathbf{g2} = -0.210422444 - 0.537639032x^2 - 0.327216587y^2 - 0.748061476xz$

$\mathbf{g3} = 0.561390307 - 0.134811931x^2 - 0.696202238y^2 + 0.426578376xz$

Check: Multiply and normalize with constant term 2

f2 = Expand[(2 * g1 * g2 * g3)/(g1[[1]] * g2[[1]] * g3[[1]])]

$2. + 3.x^2 - 5.x^4 + 1.x^6 - 3.y^2 - 16.x^2y^2 + 8.x^4y^2 - 5.y^4 + 16.x^2y^4 + 7.y^6 + 9.xz - 4.x^3z - 2.x^5z - 22.xy^2z + \mathbf{5.44 * 10^{-9}x^3y^2z} + 11.xy^4z + 7.x^2z^2 - 4.x^4z^2 - 11.x^2y^2z^2 + 1.x^3z^3$

Difference from original in bold, 2-Norm of error about $5 * 10^{-8}$.

Blow up of Curve Singularity

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Consider the plane curve

$$f = 8x^3 + x^4 + 12x^2y - 20xy^2 - x^2y^2 + 4y^3 + y^4$$

with singularity at origin. Since we don't a priori know the tangents to the branches we blow up using the random quadratic transformation

$$z = \frac{-0.292846x + 0.999554y}{0.763056x + 0.963694y}$$

Thus we have the system $\{f, g\}$ with

$$g = -0.292846x + 0.999554y - 0.763056xz - 0.963694yz$$

The solution curve has two components, one defined by $x = 0, y = 0$ has multiplicity 3, the desingularized curve we want is the other component.

Intersecting curve with random hyperplane gives 8 points, we reject 3 in the unwanted component and calculate $\mathcal{D}_6(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_5\})$ for the other 5 points using tolerance $\varepsilon = 10^{-12}$.

Blowing up (continued)

Timing[G = GDiff[{f, g}, P, 6, X, eps];]
$\{4.44, Null\}$

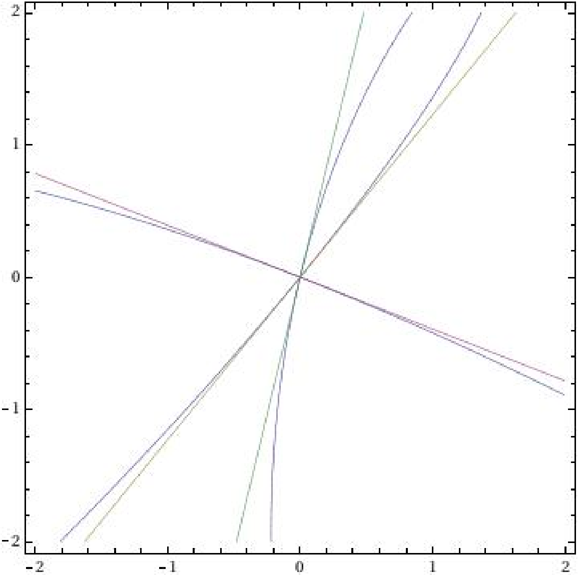
We next find a minimal H-basis

Timing[B = MBasis1[G, 6, X, eps];]
$\text{FinalTolerance} = 7.74628 * 10^{-6}$
$\text{Degrees } \{2, 4, 4, 4, 4\}$
$\text{AffineHilbertFunction } \{1, 4, 9, 16, 21, 26, 31\}$
$\{1.41, \text{Null}\}$

which gives a curve of degree 5.
To find the points of this curve over $(0, 0)$ we evaluate all 5 members of the H-basis at $x = 0, y = 0$, then solve for z. The common solution is, each after rounding to final tolerance above,

$$\{-1.7727, 0.479915, 0.810186\}$$

Calculating the Jacobian at the points $(0, 0, -1.7727)$ etc. we find these are non-singular points, and we can project the tangent lines to $-0.35787x - 0.914374y,\ 0.775099x - 0.631633y,\ 0.97079x - 0.233125y$



Curve (blue) near $(0, 0)$ with tangent lines.

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