Module Structures on the Hochschild and Cyclic Homology of Graded Rings

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1 Introduction

Module structures on various cohomology and K-theory groups have proved useful in performing calculations (see [Bloch, D, DW, GRW, LR, S-TP, S-OE, SvdK, W-MVS, W-MGR, WG]). In this paper we will show how to define module structures on Hochschild homology, Cyclic homology and Kähler differentials for certain graded rings.

In this paper $\mathbf{k}$ will be a commutative ring (usually a field or $\mathbf{Z}$), R will be a commutative $\mathbf{k}$–algebra and Λ a unitary associative R–algebra. We will write $HH_*(\Lambda)$ for the Hochschild homology group $HH_*^{\mathrm{k}}(\Lambda;\Lambda)$ and $HC_*(\Lambda) = HC_*^{\mathrm{k}}(\Lambda)$ for the Cyclic homology group of the $\mathbf{k}$–algebra Λ. $\Omega^n_{\Lambda/\mathrm{k}}$ is the module of Kähler differentials, $H^n_{dR}(\Lambda)$ is the classical deRahm cohomology and $HH^n_{dR}(\Lambda)$ is the non-commutative deRahm cohomology. Finally, $W(R)$ is the ring of Witt vectors over R, whose underlying abelian group is $1 + tR[[t]]$.

Theorem 1.1 *Let* $\mathbf{k}$ *be a commutative ring,* R *a commutative* $\mathbf{k}$*–algebra and* $\Lambda = R \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ *a unitary associative graded* R*–algebra. Then the following have natural* $W(R)$*–module stuctures.*

- $\Omega^n_{\Lambda/\mathrm{k}}/\Omega^n_{R/\mathrm{k}}$ *if* Λ *is commutative.*
- $H^n_{dR}(\Lambda)/H^n_{dR}(R)$ *if* Λ *is commutative.*
- $HH_n(\Lambda)/HH_n(R)$

- $HH^n_{dR}(\Lambda)/HH^n_{dR}(R)$
- $HC_1(\Lambda)/HC_1(R)$
- $HC_n(\Lambda)/HC_n(R)$ *if* $\mathbf{Q} \subseteq \mathbf{k}$ *or* $R = \mathbf{k}$.

Our technique, following [W-MGR], is to deduce the general graded case from the special case $\Lambda[x]$ where Λ is concentrated in degree 0 and x has degree 1.

Theorem 1.2 *Let* $\mathbf{k}$ *be a commutative ring,* R *a* $\mathbf{k}$*–algebra and* Λ *a* R*–algebra. Then the following have natural* $W(R)$*–module structures. In fact they are Carf(R)–modules.*

- $\Omega^n_{\Lambda[x]/\mathrm{k}}/\Omega^n_{\Lambda/\mathrm{k}}$ *if* Λ *is commutative.*
- $H^n_{dR}(\Lambda[x])/H^n_{dR}(\Lambda)$ *if* Λ *is commutative.*
- $HH_n(\Lambda[x])/HH_n(\Lambda)$
- $HH^n_{dR}(\Lambda[x])/HH^n_{dR}(\Lambda)$
- $HC_1(\Lambda[x])/HC_1(\Lambda)$
- $HC_n(\Lambda[x])/HC_n(\Lambda)$ *if* $\mathbf{Q} \subseteq \mathbf{k}$ *or if* $R = \mathbf{k}$.

Here *Carf(R)* is a subring of the Cartier ring *Cart(R)* of natural endomorphisms of $W(R)$. The elements of *Carf(R)* and *Cart(R)* are sums of the form $\sum V_m[r_{mn}]F_n$ where V_m is the Verschiebung, $[r]$ is the homothety and F_m is the Frobenius operator. We prove that the groups mentioned in Theorem 1.2 are *Carf(R)*–modules by using decomposition theorems for the functors Ω^*, HH_* and HC_* on polynomial extensions. Since $W(R)$ is the subring of *Carf(R)* consisting of elements $\sum V_m[r_m]F_m$ it then follows that these groups are $W(R)$–modules.

It should be noted, however, that having a *Carf(R)*–module structure is a much stronger property than having a $W(R)$ module structure. For example, for a general graded ring the groups in 1.1 are $W(R)$–modules but not *Carf(R)*–modules. As another example, if $\mathbf{Q} \subseteq \mathbf{k}$ then the decomposition of $HC_n(\Lambda[x])$ given by [K-CCM] follows easily from the fact that $HC_n(\Lambda[x])$ is a *Carf(R)*–module.

The $W(R)$–module operations on Hochschild and cyclic homology are compatible with the module operations on K-theory.

Theorem 1.3 *Let* $\mathbf{k}$*,* R *and* Λ *be as in Theorem 1.1. The Dennis trace map* $D : K_n(\Lambda)/K_n(R) \to HH_n(\Lambda)/HH_n(R)$ *is a* $W(R)$*–module homomorphism. Further if* $\mathbf{Q} \subseteq \mathbf{k}$ *then* D *factors through* $B : HC_{n-1}(\Lambda)/HC_{n-1}(R) \to HH_n(\Lambda)/HH_n(R)$ *as a* $W(R)$*–module homomorphism. If* Λ *is a polynomial ring the Dennis trace map is a linear map of Carf(R)–modules.*

If $\mathbf{Q} \subseteq \mathbf{k}$ then $W(R)$ contains R as a subring so a $W(R)$–module is also an R–module. However the R–module structures on the the groups in Theorem 1.1 and Theorem 1.2 induced from $W(R)$ will generally be different from the expected R–module structure. For example, for Kähler differentials we have the following explict description of the $W(R)$ and R–module structures.

Theorem 1.4 *If $\Lambda = R \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ is a commutative graded algebra over R then $\Omega^n_{\Lambda/\mathrm{k}}/\Omega^n_{R/\mathrm{k}}$ is a $W(R)$–module where the operation is given as follows. Let $a_0, a_1, \ldots, a_n$ be homogeneous elements of A. Let $\ell = \deg a_0 + \deg a_1 + \cdots + \deg a_n > 0$ and let $\omega = a_0 da_1 \wedge da_2 \wedge \cdots \wedge da_n$. Then*

$$(1 - rt^m) * \omega = mr^{\ell/m}\omega + r^{\ell/m-1} a_0 \sum_{i=1}^{n} (\deg a_i) a_i da_1 \wedge \cdots \wedge da_{i-1} \wedge dr \wedge da_{i+1} \wedge \cdots \wedge da_n$$

if $m|\ell$ and $(1 - rt^m) * \omega = 0$ otherwise. Furthermore, if $\mathbf{k}$ contains $\mathbf{Q}$ then $\Omega^n_\Lambda/\Omega^n_R$ has an R–module structure induced by $W(R)$ satisfying (for ω above)

$$r * \omega = r\omega + a_0 \sum_{i=1}^{n} \left(\frac{\deg a_i}{\ell}\right) a_i da_1 \wedge \cdots \wedge da_{i-1} \wedge dr \wedge da_{i+1} \wedge \cdots \wedge da_n$$

It should be noted that any graded R–module $A = A_1 \oplus A_2 \oplus \cdots$ has a $W(R)$–module structure given by (for $a_i \in A_i$)

$$(1 - rt^m) * a_i = \begin{cases} mr^{i/m} a_i & \text{if } m|i \\ 0 & \text{otherwise} \end{cases}$$

We call this $W(R)$–module structure the structure induced by "restriction of scalars along the ghost map." The formulas in 1.4 show that the $W(R)$–module structure on $\Omega^n_\Lambda/\Omega^n_R$ is not induced by the ghost map in general.

As an application of our results we improve and correct the computation of the K-theory of a union of lines through the origin of affine m-space $\mathbf{A}^m_{\mathbf{k}}$ contained in [W-LL]. For a functor F : *Graded Rings* $\to$ *Abelian groups* we write $\tilde{F}(A) = \ker(F(A) \to F(A_0)) \approx F(A)/F(A_0)$ where A_0 is the degree zero part of A. We then have

Theorem 1.5 *Let A be the coordinate ring of $b+1$ lines through the origin of $\mathbf{A}^m_{\mathbf{k}}$ where $\mathbf{k}$ is a field of characteristic 0. Let C be the seminormalization of A. Then there are $W(\mathbf{k})$-modules $\mathcal{V}_n$, $\mathcal{W}_n$ and exact sequences of $W(\mathbf{k})$-modules*

$$0 \to \mathcal{V}_{n+1} \to C/A \otimes \Omega^n_{\mathbf{k}} \to \tilde{K}_n(A) \to \widetilde{HC}_{n-1}(A) \to \mathcal{W}_n \to 0$$

and

$$0 \to \tilde{A} \otimes \Omega^{n-1}_{\mathbf{k}} \to \mathcal{W}_n \to \mathcal{V}_n \to 0$$

Section 2 contains a discussion of the rings $W(R)$, *Cart(R)* and *Carf(R)* and modules over these rings. In Section 3 we discuss the Carf(R) and $W(R)$–module structures on Kähler differentials, using an explict description of the operations. Up to here no knowledge of cyclic or Hochschild homology is assumed of the reader. In Section 4 we construct the Verschiebung, homothety and Frobenius operators and show that they define *Carf(R)*–module structures on $NHH_*(\Lambda)/NHH_*(R)$ and, in case $\mathbf{Q} \subseteq \mathbf{k}$, on $NHC_*(\Lambda)/NHC_*(R)$. Section 5 treats the case of graded R–algebras and in Section 6 we prove, without the assumption that $\mathbf{k}$ contains $\mathbf{Q}$, that $NHC_*(\Lambda)/NHC_*(R)$ is at least a *Carf(*$\mathbf{k}$*)*–module.

2 $W(R)$ and Carf(R) - modules

Our method of constructing $W(R)$–modules is based on Cartier's method [C1, C2] of constructing a $W(R)$–module structure on the set C of curves in a formal group; he embeds $W(R)$ in a larger ring *Cart(R)* which acts naturally on C. We are going to use an intermediate ring *Carf(R)* because *Cart(R)* is too big for our purposes. In order to fix notation, we first recall some well-known facts about $W(R)$ and *Cart(R)*.

2.1 Witt Vectors

Let R be a commutative ring. The ring $W(R)$ of (big) Witt vectors over R is a topologically complete commutative ring. Since we know of no definitive literature for $W(R)$, we begin with an elementary description of $W(R)$[1]. Its underlying topological abelian group is the subgroup $(1+tR[[t]])^*$ of units of the power series ring $R[[t]]$, the topology being induced from the t-adic topology on $R[[t]]$. The multiplicative unit of $W(R)$ is $(1-t)$. Using $*$ for the ring product, the multiplicative structure of $W(R)$ is completely determined by the formula

$$(1-rt^m) * (1-st^n) = (1-r^{n/d}s^{m/d}t^{mn/d})^d, \; d = \gcd(m,n) \tag{1}$$

This is because every element $w \in W(R)$ has a unique convergent expansion $w(t) = \prod_{m=1}^{\infty}(1-r_m t^m)$ with $r_m \in R$. For historical reasons the r_m are called the *Witt coordinates* of $w(t)$ and $w(t)$ is sometimes represented as the sequence of coordinates $(r_1, r_2, \ldots)$. See [Lang] for this point of view; for another interpretation of the multiplication in $W(R)$ see [Grays].

Here are some basic remarks about the ring structure of $W(R)$. Let p be a prime number. $W(\mathbf{F}_p)$ is a countable product of copies of $\hat{\mathbf{Z}}_p$, so if $p = 0$ in R then $W(R)$

[1] $W(R)$ differs somewhat from the ring $W^{(p)}(R)$ first used by E. Witt in 1936 in Kummer theory, and later by I.S. Cohen in 1946 in ring theory and Dieudonné in 1955 in Lie theory, all in the case when R is a perfect field of characteristic p. The formulation we give appeared in P. Cartier's 1967 paper [C1]. The authors refer the reader to [Bloch, Bour, Grays, Haz, Lang, Laz, W-MVS] for more details.

is a $\hat{\mathbf{Z}}_p$–algebra. In contrast, if $\frac{1}{p} \in R$ then $\frac{1}{p} \in W(R)$ because $(1-t)^{1/p} \in 1 + tR[[t]]$. If $\mathbf{Q} \subseteq R$ then not only is $\mathbf{Q} \subseteq W(R)$ but $W(R)$ isomorphic to the R–algebra $\prod_{i=1}^{\infty} R$ via the ghost map (below) and the diagonal ring homomorphism $\lambda_t : R \to W(R)$ is given by the power series expansion

$$\lambda_t(r) = (1-t)^r = 1 - rt + \binom{r}{2} t^2 + \cdots + (-1)^i \binom{r}{i} t^i + \cdots$$

For any commutative ring R there is a ring homomorphism $gh : W(R) \to \prod_{i=1}^{\infty} R$ called the *ghost map*; the n^{th} component of gh is the map $gh_n : W(R) \to R$ defined by

$$gh_n(\prod(1 - r_d t^d)) = \sum_{d|n} d r_d^{n/d}.$$

See for example [Grays, p. 253], [Lang, p. 234]. The ghost map is sometimes defined by identifying $\prod_{i=1}^{\infty} R$ with the R–module $\Omega(R[[t]]/R)$ of continuous Kähler differentials (the i^{th} coordinate idempotent e_i corresponds to $t^{i-1}dt$, see Example 2.7 below), via the formula [Bour, Ex.1.42]

$$\sum gh_i(w) t^{i-1} dt = -\frac{d}{dt} \log(w) dt.$$

From this formula it is easy to see that if R is a $\mathbf{Q}$–algebra then gh is an isomorphism and that $gh(\lambda_t) : R \to \prod R$ is the diagonal inclusion, i.e., $gh(\lambda_t(r)) = (r, r, r, \dots)$ The following lemma/definition is an easy exercise.

Lemma 2.1 *Let A be an R–module and $n \geq 1$. Write $At^n = \{at^n : a \in A\}$ for the $W(R)$–module induced from A by restriction of scalars by $gh_n : W(R) \to R$. Then the $W(R)$–module structure is determined by the formula:*

$$(1 - r_d t^d) * at^n = \begin{cases} d r_d^{n/d} at^n & \text{if } d|n \\ 0 & \text{otherwise} \end{cases}$$

If R is a $\mathbf{Q}$–algebra, the restriction of scalars along $\lambda_t : R \to W(R)$ recovers the original R–module structure on A.

Definition 2.2 Let A be an R–module. We shall write $tA[t]$ and $tA[[t]]$ for the direct sum $\coprod_{i=1}^{\infty} At^i$ and the direct product $\prod_{i=1}^{\infty} At^i$ of the $W(R)$–modules At^i defined in 2.1 above. We shall write elements of these modules as sums $\sum a_i t^i$. If $r \in R$ we can also define a $W(R)$–module endomorphism [r] on At^n, $tA[t]$ and $tA[[t]]$, called the *homothety*, by the formula $[r](at^i) = ar^i t^i$.

We will call a $W(R)$–module M *continuous* if it satisfies the rule that for every $v \in M$, $\mathrm{ann}_{W(R)}(v)$ is an open ideal in $W(R)$. That is, there is a k such that $(1-rt)^m * v = 0$ for all $r \in R$ and all $m \geq k$. Note that $tA[t]$ is a continuous $W(R)$–module but that $tA[[t]]$ is not.

Example 2.3 Let $M_1, M_2, \dots$ be R-modules. Then $\bigoplus M = M_1 x \oplus M_2 x^2 \oplus M_3 x^3 \cdots$ is a continuous $W(R)$–module. This is because the category of continuous $W(R)$–modules is closed under direct sums (and every colimit for that matter). The individual $M_i x^i$ are continuous because if $d > i$ then $(1 - rt^d) * M_i x^i = 0$.

We note that if R is of characteristic p then any continuous $W(R)$–module is a p-group. This comes from the fact that $p^n = (1-t)^{p^n} = (1 - t^{p^n})$.

When $\mathbf{Q} \subseteq R$ it is well-known that $gh : W(R) \to \prod_{i=1}^{\infty} R$ is a ring isomorphism. The element $\varepsilon_i = exp(-t^i/i)$ of $W(R)$ is idempotent and corresponds to the i^{th} coordinate idempotent of $\prod_{i=1}^{\infty} R$.

Theorem 2.4 *Let $\mathbf{Q} \subseteq R$ and M be a continuous $W(R)$–module. Setting $M_i = \varepsilon_i M$, M is isomorphic to the module $\bigoplus M_i x^i$ of 2.3. Moreover, for each i M_i is an eigenspace with eigenvalue r^i for each of the endomorphisms $(1-rt)*$ of M. In particular if $r \in \mathbf{Q}$, $r \notin \{\pm 1, 0\}$, this is the eigenspace decomposition of M for the operator $(1-rt)*$.*

Proof As $w * \varepsilon_i = gh_i(w)\varepsilon_i$ in $\varepsilon_i W(R) \approx R$ it is clear that $M_i = M_i x^i$. Since $\varepsilon_i * \varepsilon_j = 0$ for $i \neq j$, the evident map $\bigoplus M_i x^i \to M$ is an injection. Since M is a continuous $W(R)$–module, given $v \in M$ we can pick k so that $\varepsilon_i * M = 0$ for $i > k$. It is then seen that $v = 1 * v = \sum_{i=1}^{k} \varepsilon_i v$ is a finite sum of elements of the submodules M_i. Finally, the assertions about eigenspaces follow from the observation that $gh_i(1-rt) = r^i$.

2.2 The rings *Cart(R)*

In [C1], Cartier constructed the non-commutative ring *Cart(R)* as follows. Consider the functor W^+ from the category of commutative R-algebras to abelian groups, sending the algebra Λ to the abelian group $(1 + t\Lambda[[t]])^*$ underlying $W(\Lambda)$. *Cart(R)* is the ring of natural endomorphisms of this functor. As such it clearly contains $W(R)$ as a subring. It also contains the homothety $[r]$, Verschiebung V_m, and Frobenius F_m maps defined as follows. For $w = w(t) \in W(R)$,

$$
\begin{aligned}
V_m(w) &= w(t^n) \\
[r](w) &= w(rt) \\
F_m(w) &= Norm\ w(\sqrt[m]{t})
\end{aligned}
$$

the norm being taken from $R[[\sqrt[m]{t}]]$ to $R[[t]]$. In particular $F_m(1-rt) = (1-r^m t)$ and F_m is actually a ring endomorphism of $W(R)$ (see [Bour, Ex.1.47]). Cartier shows in [C1] that every element $u \in$ *Cart(R)* has a unique convergent row-finite expansion $u = \sum V_m[r_{mn}]F_n$ for $m, n \geq 1$. "Row-finite" refers to the matrix (r_{mn}): for each m only finitely many of the r_{mn} are non-zero. For example, the inclusion $W(R) \subseteq$ *Cart(R)* is determined by the formula

$$\prod_{m=1}^{\infty}(1-r_m t^m) = \sum_{m=1}^{\infty} V_m[r_m]F_m.$$

(The reader should beware that the *Cart(R)*–module operations on $W(R)$ described above are not left multiplication.)

In order to give a presentation for *Cart(R)*, recall [AT, p. 258] that there are integer polynomials $p_d(x,y)$ of degree d in x and y, such that

$$x^m + y^m = \sum_{d|m} d p_d(x,y)^{m/d}$$

for each $m = 1, 2, \cdots$. They may be determined recursively. For instance,

$$p_1(x,y) = x+y \quad , \quad p_2(x,y) = -xy$$
$$p_3(x,y) = -(x^2y + xy^2) \quad , \quad p_4(x,y) = -(x^3y + 2x^2y^2 + xy^3).$$

Cartier also showed in [C2] (see also [Haz, Laz]) that the ring *Cart(R)* satisfies the following identities, which completely determine the ring structure on *Cart(R)*:

Theorem 2.5 (Cartier's Identities) *Let R be a commutative ring. The following identities hold in Cart(R) for all $m, k \geq 1$ and all $r, s \in R$.*

(i) $[1] = V_1 = F_1$ *is the multiplicative identity.*
(ii) $[r][s] = [rs]$, $V_kV_m = V_{km}$, $F_kF_m = F_{km}$
(iii) $[r]V_m = V_m[r^m]$, $F_m[r] = [r^m]F_m$
(iv) F_mV_m *is multiplication by* m
(v) *If* $(k,m) = 1$ *then* $V_mF_k = F_kV_m$
(vi) $[r] + [s] = \sum_{m\geq 1} V_m[p_m(r,s)]F_m$

It should be noted that identities (ii), (iv) and (v) together say that $F_kV_m = dV_{m/d}F_{k/d}$ where $d = (m,k)$. In particular,

$$F_k(1-rt^m) = dV_{m/d}[r^{k/d}]F_{mk/d} = (1-r^{k/d}t^{m/d})^d F_{k/d}$$

Thus $W(R)$ is not in the center of *Cart(R)*.

Any *Cart(R)*–module will be a $W(R)$–module, but *Cart(R)*–modules tend to be too big for our purposes. For example, $W(R) = 1 + tR[[t]]$ is a cyclic *Cart(R)*–module by definition. Using the ghost map, we can construct the following family of *Cart(R)*–modules.

Example 2.6 If A is any R–module, the $W(R)$–module structure on $tA[[t]]$ described in 2.2 extends to a left *Cart(R)*–module structure via the formulas

$$\begin{aligned} V_m(at^i) &= at^{mi} \\ [r](at^i) &= ar^i t^i \\ F_m(at^i) &= \begin{cases} mat^{i/m} & \text{if } m|i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We leave the verification of these facts to the reader, noting that since $F_n(at^i) = 0$ whenever $n > i$ the formula for $\sum V_m[r_{mn}]F_n$ is well defined.

Note that if Λ is a commutative R–algebra containing $\mathbf{Q}$ the logarithm $W(\Lambda) \to t\Lambda[[t]$ is an isomorphism of *Cart(R)*–modules.

Example 2.7 [Bloch, p. 195] Let A be an R–module and write $\Omega(A[[t]]/A)$ for the product $\prod_{i=1}^{\infty} Ae_i$ of copies of A. (The generators e_i are often written as $t^{i-1}dt$.) Define operations V_m, $[r]$, F_m by the formulas

$$\begin{aligned} V_m(ae_i) &= mae_{mi} \\ [r](ae_i) &= ar^i e_i \\ F_m(ae_i) &= \begin{cases} ae_{i/m} & \text{if } m|i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Because of the row-finiteness condition on *Cart(R)*, the formula

$$(\sum_{m,n} V_m[r_{mn}]F_n)(\sum_i a_i e_i) = \sum_{m,n,i} m r_{mn}^i a_{in} e_{mi} = \sum_k \sum_{m|k} \sum_n m r_{mn}^{k/m} a_{kn/m} e_k$$

is well defined. By direct observation, Cartier's identities 2.5 are satisfied. Therefore $\Omega(A[[t]]/A)$ is a *Cart(R)*–module.

Any *Cart(R)*–module has an underlying $W(R)$–module structure via the inclusion $W(R) \subset$ *Cart(R)*. The two distinct *Cart(R)*–modules $\Omega(A[[t]]/A)$ and $tA[[t]]$ have isomorphic underlying $W(R)$–module structures, via the correspondence $e_i \leftrightarrow t^i$.

Using the explicit formulas for $V_m, [r]$, and F_m on $W(R)$ one checks (as in [Bloch]) that the ghost map $gh : W(\Lambda) \to \Omega(\Lambda[[t]]/\Lambda)$ is a *Cart(R)*–module homomorphism whenever Λ is a commutative R–algebra. Of course, if $\mathbf{Q} \subseteq R$ it is an isomorphism, and the formulas for $V_m, [r]$, and F_m on $\Omega(\Lambda[[t]]/\Lambda)$ are obtained by transferring the formulas for $W(\Lambda)$.

2.3 The ring *Carf(R)*

We define *Carf(R)* to be the subset of *Cart(R)* consisting of elements $\sum V_m[r_{mn}]F_n$ in which the matrix (r_{mn}) is both row and column-finite. That is, for each n only finitely many of the r_{mn} are non-zero. Since *Carf(R)* contains $W(R)$ and hence the element "–1" $= (1-t)^{-1}$, it is easy to see from Cartier's identities 2.5 that *Carf(R)* is a subring of *Cart(R)* containing $W(R)$.

The ring *Carf(R)* is a topological ring in which the left ideals $I_N = \{\sum V_m[r_{mn}]F_n : n \geq N\}$ form a basic family of neighborhoods of 0, and $W(R) \subset$ *Carf(R)* is an inclusion of topological rings. We say that a left *Carf(R)*–module A is *continuous* provided that for every $a \in A$ $ann_{Carf(R)}(a)$ is an open left ideal of *Carf(R)*, i.e. $F_n(a) = 0$ for all but finitely many n. By restriction of scalars, a continuous *Carf(R)*–module is also a continuous $W(R)$–module.

Theorem 2.8 *Suppose that A is an abelian group equipped with endomorphisms $F_m, [r]$ and V_m, $(m \geq 1,\ r \in R)$ satisfying Cartier's identities 2.5 (i) — (vi). Suppose also for each $a \in A$*
(vii) $F_n(a) = 0$ *for all but finitely many n*
Then these operations make A into a continuous Carf(R)*–module, and a fortiori into a continuous $W(R)$–module.*

The proof of the above Theorem is straightforward, once we notice that each element of *Carf(R)* has a unique representation of the form $\sum V_m[r_{mn}]F_n$ which is both row and column-finite, and that only finitely many of the terms $V_m[r_{mn}]F_n$ are non-zero on any given $a \in A$.

Example 2.9 Let A be an R–module, then defining $V_m, [r]$ and F_m on $tA[t]$ by the formulas of 2.6 satisfies the identities of 2.8 and thus $tA[t]$ is a continuous *Carf(R)*–module.

Example 2.10 (Differentials) Let A be an R–module. We shall write $\Omega(A[t]/A)$ for the direct sum $\coprod_{i=1}^{\infty} Ae_i$ of copies of A on generators e_i. (The generators e_i are often written as $t^{i-1}dt$. This is a subgroup of the *Cart(R)*–module $\Omega(A[[t]]/A) = \prod Ae_i$ of 2.7, and is closed under the operations $V_m, [r]$ and F_m of loc. cit. Since Cartier's identities hold in $\Omega(A[[t]]/A)$ and 2.8(vii) is clearly satisfied, 2.8 shows that $\Omega(A[t]/A)$ is a continuous (left) *Carf(R)*–module. Our notation comes from the fact that if Λ is a commutative ring and we write $t^{i-1}dt$ for e_i then $\Omega(\Lambda[t]/\Lambda) \approx \Omega_{\Lambda[t]/\Lambda}$ is the usual Λ–module of Kähler differentials of the polynomial ring $\Lambda[t]$ over Λ.

Note that the underlying $W(R)$–modules $tA[t]$ of 2.9 and $\Omega(A[t]/A)$ of 2.10 are isomorphic via the correspondence $t^i \mapsto e_i$. If R contains $\mathbf{Q}$ then $tA[t] \approx \Omega(A[t]/A)$ as *Carf(R)*–modules via the correspondence $t^i \mapsto ie_i$.

Example 2.11 Define the de Rham differential $D : tA[t] \to \Omega(A[t]/A)$ by the formula $D(t^i) = it^{i-1}dt = ie_i$. This is a *Carf(R)*–module homomorphism for every R–module A. Therefore the kernel and cokernel are continuous *Carf(R)*–modules. When Λ is a commutative ring, the kernel is the de Rahm cohomology group $H^0_{dR}(\Lambda[t]/\Lambda)$ and the cokernel is the cyclic homology group $HC_1^{\Lambda}(\Lambda[t])$. (In both cases we are taking the ground ring $\mathbf{k} = \Lambda$. We will discuss both of these groups in more detail later in this paper). For example, the following are continuous *Carf*($\mathbf{Z}$)–modules:

- $HC_1(\mathbf{Z}[t]) = \mathbf{Z}/2\mathbf{Z}e_2 \oplus \mathbf{Z}/3\mathbf{Z}e_3 \oplus \cdots = \coprod_{\ell\geq 2}^{\infty} \mathbf{Z}/\ell\mathbf{Z}e_\ell$;
- $HC_1(\mathbf{Z}/\mathbf{p}[t]) = \mathbf{Z}/\mathbf{p}\mathbf{Z}e_{\mathbf{p}} \oplus \mathbf{Z}/\mathbf{p}\mathbf{Z}e_{2\mathbf{p}} \oplus \cdots = \coprod_{\ell=1}^{\infty} \mathbf{Z}/\mathbf{p}\mathbf{Z}e_{\ell\mathbf{p}}$;
- $H^0_{dR}(\mathbf{Z}/\mathbf{p}[t] \ / \ \mathbf{Z}/\mathbf{p}\mathbf{Z}) = t^{\mathbf{p}}\mathbf{Z}/\mathbf{p}\mathbf{Z}[t^{\mathbf{p}}]$.

We end this section with a characterization of *Carf(R)*–modules when $\mathbf{Q} \subseteq R$.

Lemma 2.12 *If $\mathbf{Q} \subseteq R$ then R is the center of both Carf(R) and Cart(R) via the inclusions $R \xrightarrow{\lambda_t} W(R) \subseteq$ Carf(R) $\subseteq$ Cart(R). In particular, V_m and F_m act R-linearly on any Carf(R)–module.*

Proof If Λ is a commutative R–algebra then a direct calculation shows that the endomorphisms $\lambda_t(r) = (r, r, \ldots)$ commute with V_m, [s] and F_m on $\prod_{i=1}^{\infty} \Lambda \approx W(\Lambda)$ (via the ghost map). By definition of *Cart(R)* this means that $\lambda_t(r)$ is in the center of *Cart(R)* for each $r \in R$. Now suppose $\Theta = \sum V_m[r_{mn}]F_n$ is in the center of *Carf(R)* or *Cart(R)*; by adding $\lambda_t(-r_{11})$ to Θ we may assume that $r_{11} = 0$. Choose m, n minimizing $m + n$ subject to $r_{mn} \neq 0$; computing the commutators $[F_m, \Theta]$ and $[\Theta, V_n]$ we obtain a contradiction unless $\Theta = 0$.

Theorem 2.13 *Let M be a continuous Carf(R)–module where $\mathbf{Q} \subseteq R$. Then $M = \coprod_{i=1}^{\infty} M_i$ where M_i is the eigenspace for r^i under the endomorphism $[r]$ and is a R-submodule. Further $V_m : M_i \to M_{mi}$ and $F_m : M_{im} \to M_i$ are R-module isomorphisms. In particular $M_i \approx M_1$ for all $i \geq 1$. Finally for $a \in M_i$ we have*

$$r * a = \frac{1}{i} V_i[r]F_i(a). \tag{2}$$

Proof $F_iV_i = i$ by the Cartier identity (iv). On the other hand, $V_iF_i = V_i[1]F_i$ is a Witt vector with $gh_i(V_iF_i) = i$ so $V_iF_i\varepsilon_i = i\varepsilon_i$ and hence V_iF_i restricted to M_i is also i. This proves the isomorphisms. In addition this shows $r * a = r * \frac{1}{i}V_iF_i(a) = \frac{1}{i}V_i(r * F_i(a)) = \frac{1}{i}V_i([r]F_i(a)) = \frac{1}{i}V_i[r]F_i(a)$ since $F_i(a) \in M_1$ which is the eigenspace of r under $[r]$.

Corollary 2.14 *Let M be a continuous left Carf(R)–module, $\mathbf{Q} \subseteq R$. Then as a Carf(R)–module M is isomorphic to the module $tA[t]$ of Example 2.9 where $A = M_1$.*

Proof Map $tA[t] \to M$ by $at^i \mapsto V_i(a)$. By the above theorem this gives an isomorphism $(tA[t])_i \to M_1 \to M_i$ of R–modules and hence an R–module isomorphism of graded R–modules $tA[t] \to M$. A direct calculation using the Cartier identities and formula (2) shows that this is a *Carf(R)*–module isomorphism.

3 Module structures on Ω^n_Λ

In this section we give an explicit brute force description of the *Carf(R)* and $W(R)$-module structures on Kähler differentials.

In this section $\mathbf{k}$ will be a commutative ring (usually a field or $\mathbf{Z}$), R and Λ will be commutative $\mathbf{k}$–algebras. We will write Ω^n_Λ for the module of Kähler differentials $\Omega^n_{\Lambda/\mathrm{k}}$ and $N\Omega^n_\Lambda = \ker(\Omega^n_{\Lambda[x]} \xrightarrow{x \mapsto 0} \Omega^n_\Lambda)$. We will let $\Omega^0_\Lambda = \Lambda$ and we note that $N\Omega^0_\Lambda = x\Lambda[x]$ has the Carf-module structure described in Example 2.9.

Let $\Lambda = R[x]$, it is well known [Mats, 26.J] that $\Omega_\Lambda = (\Lambda \otimes_R \Omega_R) \oplus \Lambda dx$. Since $\Omega^2_{\mathrm{k}[x]} = 0$ we have $\Omega^n_\Lambda = (\Lambda \otimes_R \Omega^n_R) \oplus (\Lambda \otimes_R \Omega^{n-1}_R \wedge dx)$ (see also [Kunz, Cor. 4.10]). Taking the kernel as $x \mapsto 0$ we have $N\Omega^n_R = xR[x]\Omega^n_R \oplus R[x]\Omega^{n-1}_R \wedge dx$.

Definition 3.1 *Let Λ be a commutative $\mathbf{k}$–algebra and $n \geq 1$. Define endomorphisms $V_m, [r]$ and F_m on $N\Omega^n_\Lambda$ by*

$$\begin{aligned}
V_m(x^i\omega) &= x^{im}\omega \\
[r](x^i\omega) &= r^i x^i \omega \\
F_m(x^i\omega) &= \begin{cases} m x^{i/m}\omega & \text{if } m|i \\ 0 & \text{otherwise} \end{cases} \\
&\text{and} \\
V_m(x^{i-1}\nu \wedge dx) &= x^{im-m}\nu \wedge d(x^m) = m x^{im-1}\nu \wedge dx \\
[r](x^{i-1}\nu \wedge dx) &= r^{i-1}x^{i-1}\nu \wedge d(rx) = r^i x^{i-1}\nu \wedge dx + r^{i-1}x^i \nu \wedge dr \\
F_m(x^{i-1}\nu \wedge dx) &= \begin{cases} x^{i/m-1}\nu \wedge dx & \text{if } m|i \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

where $r \in \Lambda,\ \omega \in \Omega^n_\Lambda$ and $\nu \in \Omega^{n-1}_\Lambda$.

Theorem 3.2 *Let Λ be a commutative $\mathbf{k}$–algebra. The above endomorphisms make $N\Omega^n_\Lambda$ a Carf(Λ)–module. This makes $N\Omega^n_\Lambda$ a $W(R)$–module by the formulas*

$$\begin{aligned}
(1 - rt^m) * x^i\omega &= m r^{i/m} x^i \omega \\
(1 - rt^m) * x^{i-1}\nu \wedge dx &= m r^{i/m} x^{i-1}\nu \wedge dx + r^{i/m-1}x^i \nu \wedge dr
\end{aligned}$$

if $m|i$ and 0 otherwise. Finally, there is a short exact sequence

$$0 \to x\Omega^n_\Lambda[x] \to N\Omega^n_\Lambda \to \Omega^{n-1}_\Lambda \otimes \Omega(\mathbf{k}[x]/\mathbf{k}) \to 0$$

of Carf(Λ)–modules, where the left and right terms have the Carf(Λ)–module structures of 2.2 and 2.10 respectively.

Remark 3.3 If $\mathbf{Q} \subseteq \mathbf{k}$ it is easily seen that if $M = N\Omega^n_\Lambda$ then M_i is generated by forms $x^i\omega$ and $x^{i-1}\nu \wedge dx$ where $\omega \in \Omega^n_\Lambda$ and $\nu \in \Omega^{n-1}_\Lambda$. However by Theorem 2.13 the Λ–module operation on M_i is given by

$$\begin{aligned} r * (x^i\omega) &= rx^i\omega \\ r * (x^{i-1}\nu \wedge dx) &= rx^{i-1}\nu \wedge dx + \frac{1}{i}x^i\nu \wedge dr \end{aligned}$$

which is not the expected operation.

Proof We must verify the Cartier identities in Theorem 2.8. Since V_m and F_m respect the direct sum decomposition, the verification of the identities involving only V_m, F_m is straightforward. The identity $[r][s] = [rs]$ is also straightforward and the identity $[r]V_m = V_m[r^m]$ is similar to the other part of identity (iii). Thus we will demonstrate only the identity $F_m[r] = [r^m]F_m$ of identity (iii) and identity (vi).

If m does not divide i then it is easily seen that $F_m[r](x^i\omega) = [r^m]F_m(x^i\omega) = 0$ and $F_m[r](\nu x^{i-1} \wedge dx) = [r^m]F_m(\nu x^{i-1} \wedge dx) = 0$. Thus we suppose that $m|i$. Calculating we have $F_m[r](x^i\omega) = F_m(r^ix^i\omega) = mr^ix^{i/m}\omega = m(r^m)^{i/m}x^{i/m}\omega = [r^m](mx^{i/m}\omega) = [r^m]F_m(x^i\omega)$. On the other summand

$$\begin{aligned} &F_m[r](x^{i-1}\nu \wedge dx) \\ &= F_m(r_ix^{i-1}\nu \wedge dx + r^{i-1}x^i\nu \wedge dr) \\ &= r^i\nu x^{i/m-1} \wedge dx + mr^{i-1}x^{i/m}\nu \wedge dr. \end{aligned}$$

On the other hand, using the differentiation formula $d(r^m) = mr^{m-1}dr$

$$\begin{aligned} &[r^m]F_m(x^i\nu \wedge dx) = [r^m](x^{i/m-1}\nu \wedge dx) \\ &= (r^m)^{i/m}x^{i/m-1}\nu \wedge dx + (r^m)^{i/m-1}x^{i/m}\nu \wedge d(r^m) \\ &= r^ix^{i/m-1}\nu \wedge dx + mr^{m-1}r^{i-m}x^{i/m} \wedge dr \\ &= r^ix^{i/m-1}\nu \wedge dx + mr^{i-1}x^{i/m} \wedge dr \end{aligned}$$

For identity (vi) we have on the first summand

$$\begin{aligned}
&\sum_{m\geq 1} V_m[p_m(r,s)]F_m(x^i\omega)\\
&= \sum_{m|i} V_m[p_m(r,s)](F_m(x^i\omega)) = \sum_{m|i} V_m[p_m(r,s)](mx^{i/m}\omega)\\
&= \sum_{m|i} V_m(mp_m(r,s)^{i/m}x^{i/m}\omega) = \sum_{m|i} mp_m(r,s)^{i/m}x^i\omega\\
&= \left(\sum_{i|m} mp_m(r,s)^{i/m}\right)x^i\omega = (r^i+s^i)x^i\omega\\
&= r^ix^i\omega + s^ix^i\omega = [r]x^i\omega + [s]x^i\omega\\
&= ([r]+[s])(x^i\omega)
\end{aligned}$$

The second summand presents added difficulties in several ways. We first consider the case where $\mathbf{Q} \subseteq \mathbf{k}$. Then as above

$$\begin{aligned}
&\sum_{m\geq 1} V_m[p_m(r,s)]F_m(x_{i-1}\nu\wedge dx)\\
&= \sum_{m\geq 1} V_m[p_m(r,s)]F_m(\frac{1}{i}\nu\wedge dx^i)\\
&= \sum_{m|i} V_m[p_m(r,s)](\frac{1}{i}\nu\wedge dx^{i/m})\\
&= \sum_{m|i} \frac{1}{i}V_m(\nu\wedge d(p_m(r,s)^{i/m}x^{i/m}))\\
&= \sum_{m|i} \frac{m}{i}\nu\wedge d(p_m(r,s)^{m/i}x^i)\\
&= \frac{1}{i}\nu\wedge d\left(\sum_{m|i} p_m(r,s)^{i/m}\right)x^i\\
&= \frac{1}{i}\nu\wedge d(r^i+s^i)x^i\\
&= \frac{1}{i}\nu\wedge dr^ix^i + \frac{1}{i}\nu\wedge ds^ix^i\\
&= \frac{1}{i}[r]\nu\wedge dx^i + \frac{1}{i}[s]\nu\wedge dx^i\\
&= [r](x^{i-1}\nu\wedge dx) + [s](x^{i-1}\nu\wedge dx)
\end{aligned}$$

$$= \quad ([r]+[s])(x^{i-1}\nu \wedge dx)$$

For the case where $\mathbf{k}$ does not contain $\mathbf{Q}$ we let $R = \mathbf{Z}[r', s']$ where r', s' are indeterminants and note that the result is true for $N\Omega^1_{R/\mathbf{Z}}$ since this is a subgroup of $N\Omega^1_{R\otimes\mathbf{Q}}$ where the formula has just been verified. Then given $x^{i-1}\nu \wedge dx = \nu \wedge x^{i-1}dx \in \Omega^n_\Lambda$

$$\begin{aligned}
&\sum_{m\geq 1} V_m[p_m(r,s)]F_m(\nu \wedge x^{i-1}dx) \\
&= \sum_{m\geq 1} \nu \wedge V_m[p_m(r',s')]F_m(x^{i-1}dx) \\
&= \nu \wedge ([r']+[s'])(x^{i-1}dx) = ([r]+[s])(x^{i-1}\nu \wedge dx)
\end{aligned}$$

This completes the proof.

Now let $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ be a commutative graded ring with $\Lambda_0 = R$ a $\mathbf{k}$–algebra. An element $a \in \Lambda_i$ is called *homogeneous of degree i*. As above we let Ω^n_Λ denote $\Omega^n_{\Lambda/\mathrm{k}}$ and we will write $\tilde{\Omega}^n_\Lambda$ for $\Omega^n_\Lambda/\Omega^n_R$. If $n = 0$ we write $\tilde{\Omega}^0_\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ which has a $W(R)$–module structure given by Example 2.9.

Theorem 3.4 *Let $\Lambda = R \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ be a commutative graded ring where R is a $\mathbf{k}$–algebra and $n \geq 1$. Then $\tilde{\Omega}^n_\Lambda$ is a $W(R)$–module where the operation is given as follows. Let $a_0, a_1, \ldots, a_n$ be homogeneous elements of Λ. Let $\ell = \deg a_0 + \deg a_1 + \cdots + \deg a_n > 0$ and let $\omega = a_0 da_1 \wedge da_2 \wedge \cdots \wedge da_n$. Then*

$$\begin{aligned}
&(1-rt^m) * \omega = r^{\ell/m} m\omega \, + \\
&\qquad r^{\ell/m-1} a_0 \sum_{i=1}^{n} (\deg a_i) a_i da_1 \wedge \cdots \wedge da_{i-1} \wedge dr \wedge da_{i+1} \wedge \cdots \wedge da_n
\end{aligned} \tag{3}$$

if $m|\ell$ and $(1-rt^m)*\omega = 0$ otherwise. Furthermore, if $\mathbf{k}$ contains $\mathbf{Q}$ then $\tilde{\Omega}^n_\Lambda$ has an R–module structure induced by $W(R)$ satisfying (for ω above)

$$r * \omega = r\omega + a_0 \sum_{i=1}^{n} \left(\frac{\deg a_i}{\ell}\right) a_i da_1 \wedge \cdots \wedge da_{i-1} \wedge dr \wedge da_{i+1} \wedge \cdots \wedge da_n \tag{4}$$

Proof We have a ring map $\iota : \Lambda \to \Lambda[x]$ given by $\iota(a) = ax^i$ for a homogeneous element of degree i. Then ι induces a group map $\iota_\bullet : \Omega^n_\Lambda \to \Omega^n_{\Lambda[x]}$. It is easily seen that $\iota_\bullet$ restricts to a map $\iota_\bullet : \tilde{\Omega}^n_\Lambda \to N\Omega^n_\Lambda$. We denote by $(1-rt^m)*$ the group endomorphism of $\tilde{\Omega}^n_\Lambda$ induced by formula (3) and by $V_m[r]F_m*$ the group endomorphism of $N\Omega^n_\Lambda$ induced by left multiplication by this Witt vector. It may be verified by direct computation, using the formulas 3.1, that the following diagram commutes.

$$\begin{array}{ccc} \tilde{\Omega}^n_\Lambda & \xrightarrow{\iota_\bullet} & N\Omega^n_\Lambda \\ {\scriptstyle (1-rt^m)*}\downarrow & & \downarrow{\scriptstyle V_m[r]F_m*} \\ \tilde{\Omega}^n_\Lambda & \xrightarrow{\iota_\bullet} & N\Omega^n_\Lambda \end{array}$$

The map $\iota_\bullet$ is a split injection whose inverse is given by the substitution $x \mapsto 1$, so we may identify $\tilde{\Omega}^n_\Lambda$ with its image under $\iota_\bullet$. But from the diagram it is clear that the image is a $W(R)$–submodule of $N\Omega^n_\Lambda$, so $\tilde{\Omega}^n_\Lambda$ is a $W(R)$–module under the operation given by formula (1). Formula (2) can similarly be checked inside $N\Omega^n_\Lambda$ using Theorem 2.13.

Exercise The reader should check directly that Equation (4) does in fact define a R–module operation on $\tilde{\Omega}^n_A$. In particular she should show that this operation is well defined and that $s * (r * \omega) = sr * \omega$.

Remark 3.5 Let R be the polynomial ring $\mathbf{k}[r]$ and $\Lambda = R[x] = \mathbf{k}[r, x]$. In $N\Omega^1_R = \tilde{\Omega}^1_\Lambda$ the $W(R)$–module structure is given by formula (1): if $m|i$ then

$$(1 - rt^m) * (x^{i-1}dx) = mr^i x^{i-1}dx + r^{i-1}x^i dr.$$

Since this is not divisible by m when $1/m \notin \mathbf{k}$, we see from 2.1 that this $W(R)$–module structure cannot be induced from a family of R–modules via the ghost maps $gh_n : W(R) \to R$.

Recall that for $n \geq 0$ the differential $d : \Omega^n_\Lambda \to \Omega^{n+1}$ is defined by $d(a\omega) = da \wedge \omega$. This defines additive morphisms $N\Omega^n_\Lambda \to N\Omega^{n+1}_A$ and, in the case that Λ is graded, $\tilde{\Omega}^n_\Lambda \to \tilde{\Omega}^{n+1}_\Lambda$. Note that the case $n = 0$ is essentially Example 2.11. A direct calculation based on the formulas 3.1 and Theorem 3.4 shows

Proposition 3.6 *Let R be a $\mathbf{k}$–algebra and Λ be a R–algebra. The differential $d : N\Omega^n_\Lambda \to N\Omega^{n+1}_\Lambda$ is a morphism of Carf(R)–modules. If Λ is graded with $\Lambda_0 = R$ then $d : \tilde{\Omega}^n \to \tilde{\Omega}^{n+1}$ is a morphism of $W(R)$ modules. Furthermore, if $\mathbf{Q} \subseteq \mathbf{k}$ and we use the R–module structures induced by $W(R)$, then the differential is an R-linear homomorphism.*

Exercise The reader is encouraged to check directly from formula (2) of Theorem 3.4 that when $\mathbf{Q} \subseteq \mathbf{k}$ then the differential is an R-linear homomorphism for the R–module structure induced by $W(R)$. This computation will illuminate the purpose of the ℓ in the denominator of (2).

Recall also that the classical deRahm cohomology $H^i_{dR}(\Lambda)$ is the cohomology of the complex

$$0 \longrightarrow \Omega^0_\Lambda \xrightarrow{d} \Omega^1_\Lambda \xrightarrow{d} \Omega^2_\Lambda \xrightarrow{d} \cdots$$

We may then define $NH^i_{dR}(\Lambda) = \ker(H^i_{dR}(\Lambda[x]) \to H^i_{dR}(\Lambda))$ and, for graded Λ, $\tilde{H}^i_{dR}(\Lambda) = \ker(\tilde{H}^i_{dR}(\Lambda) \to \tilde{H}^i_{dR}(\Lambda_0))$.

It should be noted that if $\mathbf{Q} \subseteq \mathbf{k}$ then it is known (see [Rob]) that $NH^i_{dR}(\Lambda) = 0$ as is $\tilde{H}^i_{dR}(\Lambda)$ if Λ is graded. Thus the above corollary is of interest only when $\mathbf{k} = \mathbf{Z}$ or is of positive characteristic.

Corollary 3.7 *With* $\mathbf{k}$*,* R *and* Λ *as above,* NH^i_{dR} *is a Carf(R)–module. Further, if* Λ *is graded with* $\Lambda_0 = R$ *then* $\tilde{H}^i_{dR}(\Lambda)$ *is a* $W(R)$*–module.*

4 The Verschiebung, homothety and Frobenius

In this section we construct the "VHF" operators V_m, F_m and $[r]$ and prove that they define a *Carf(R)*–module structure on $NHH_*(\Lambda)$ for every (not necessarily commutative) R–algebra Λ. If $\mathbf{Q} \subseteq \mathbf{k}$, we deduce that the VHF operators also define a *Carf(R)*–module structure on $NHC_*(\Lambda)$. As always, R is a $\mathbf{k}$–algebra, and we are taking Hochschild and cyclic homology over $\mathbf{k}$.

We shall need a certain map $\imath = \imath_m$ from $\Lambda[x]$ into $M_m(\Lambda[x])$, the ring of $m \times m$ matrices with entries in $\Lambda[x]$. One way to describe this map is that $\imath$ imbeds Λ into the diagonal of $M_m(\Lambda[x])$ and sends x to the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & x \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Note that under $\imath$, x^m is sent to the diagonal matrix with diagonal entries x, so in a sense, x is being sent to its m-th root. Also note that the nonzero entries of $\imath(x)$ lie just below the diagonal; this includes x if we index the rows and columns modulo m.

Alternatively we can describe $\imath$ as follows. We set $y = x^m$ and regard $\Lambda[x]$ as the free $\Lambda[y]$–algebra on basis $\{x^1, x^2, \ldots, x^{m-1}\}$. Then left multiplication of $\Lambda[x]$ by an element corresponds to an endomorphism of the free $\Lambda[y]$–module on m generators, i.e. to an $m \times m$ matrix. Renaming the dummy variable x instead of y, $\imath$ sends $\Lambda[x]$ to $M_m(\Lambda[x])$.

Now let Λ be a R–algebra and recall that we are writing $HH_*(\Lambda) = HH^{\mathrm{k}}_*(\Lambda, \Lambda)$ and $HC_*(\Lambda) = HC^{\mathrm{k}}_*(\Lambda)$. Recall that these are the HH_* and HC_* of the bar complex $C_\bullet(\Lambda)$, considered as a mixed complex in the sense of [K-CCM].

Let $NC_\bullet(\Lambda)$ denote the kernel of the map $C_\bullet(\Lambda[x]) \xrightarrow{x\mapsto 0} C_\bullet(\Lambda)$. By naturality the Hochschild and cyclic homology of $NC_\bullet(\Lambda)$ is $NHH_*(\Lambda)$ and $NHC_*(\Lambda)$.

Since Λ is Morita equivalent to $M_m(\Lambda)$ there is a natural map of mixed complexes (anachronistically due to R.K. Dennis [DI, 3.7]) $C_\bullet(M_m(\Lambda)) \xrightarrow{\text{trace}} C_\bullet(\Lambda)$ given by the formula

$$g^0[g^1|g^2|\cdots|g^n] \mapsto \sum_{i_0,\ldots,i_n=1}^{n} g^0_{i_0i_1}[g^1_{i_1i_2}|\cdots|g^n_{i_ni_0}] \tag{5}$$

where $g^0, g^1, \ldots, g^n \in M_m(\Lambda)$. The trace map induces isomorphisms $HH_*(M_m(\Lambda)) \approx HH_*(\Lambda)$, $HC_*(M_m(\Lambda)) \approx HC_*(\Lambda)$, etc. We remark that *trace* is an inverse to the map $C_\bullet(\Lambda) \to C_\bullet(M_m(\Lambda))$ induced by sending an element $f \in \Lambda$ to the $m\times m$ matrix with f in the first row, first column and zeros elsewhere.

Definition 4.1 (VHF Operators) Let Λ be a $\mathbf{k}$–algebra, R be a commutative $\mathbf{k}$–algebra contained in Λ.

- The *Verschiebung* V_m is the endomorphism of the mixed complexes $C_\bullet(\Lambda[x])$ and $NC_\bullet(\Lambda)$ induced by the map $\Lambda[x] \to \Lambda[x]$ sending x to x^m.
- For $r \in R$ the *homothety* $[r]$ is the endomorphism of the mixed complexes $C_\bullet(\Lambda[x])$ and $NC_\bullet(\Lambda)$ induced by the map $\Lambda[x] \to \Lambda[x]$ sending x to rx.
- The *Frobenius* F_m is the composition $C_\bullet(\Lambda[x]) \xrightarrow{\imath_*} C_\bullet(M_m(\Lambda[x])) \xrightarrow{\text{trace}} C_\bullet(\Lambda[x])$, considered as an endomorphism of the mixed complexes $C_\bullet(\Lambda[x])$ and $NC_\bullet(\Lambda)$. That is, F_m is the transfer map associated with the ring map sending x to x^m.

The induced endomorphisms of $HH_*(\Lambda[x]), NHH_*(\Lambda), HC_*(\Lambda[x]), NHC_*(\Lambda)$, etc. will also be called $V_m, [r]$ and F_m. We remark that there is a natural decomposition $C_\bullet(\Lambda[x]) = C_\bullet(\Lambda) \oplus NC_\bullet(\Lambda)$; on the subcomplex $C_\bullet(\Lambda)$ the maps V_m and $[r]$ are the identity, while F_m is multiplication by m. The interesting thing about these "VHF" operators is their action upon NHH_* and NHC_*.

Theorem 4.2 *Let* $\mathbf{k}$ *be a commutative ring,* R *a commutative* $\mathbf{k}$*–algebra and* Λ *an* R*–algebra. Then the VHF operators* $V_m, [r]$ *and* F_m *induce a* Carf(R)*–module structure on* $NHH_n(\Lambda)$*. Moreover, there is a short exact sequence of* Carf(R)*–modules:*

$$0 \to HH_n(\Lambda) \otimes_{\mathrm{k}} x\mathbf{k}[x] \to NHH_n(\Lambda) \to HH_{n-1}(\Lambda) \otimes_{\mathbf{k}} \Omega(\mathbf{k}[x]/\mathbf{k}) \to 0$$

Here the left and right terms have the Carf(R)*–module structures given in 2.9 and 2.10, respectively. This sequence is split by the Künneth formula, but is not split as a sequence of* Carf(R)*–modules (or even as* $W(R)$*–modules) when* $\Omega_{R/\mathrm{k}}$ *is non-zero.*

To prove the preceeding theorem, we will need explicit formulas for the endomorphisms $V_m, [r]$, and F_m on $HH_n(\Lambda[x])$. For this, we need to decompose $HH_*(\Lambda[x])$ using the shuffle product and the Künneth Formula. These give the following direct sum decomposition of $HH_n(\Lambda[x])$, which is well-known [Mac, X, Thm 7.4] or [CE] when $\mathbf{k}$ is a field.

Proposition 4.3 (Künneth Formula) *Let $\mathbf{k}$ be a commutative ring, Λ any $\mathbf{k}$–algebra. Then the shuffle product $\# : C_\bullet(\Lambda) \otimes C_\bullet(\mathbf{k}[x]) \to C_\bullet(\Lambda[x])$ is a quasi-isomorphism, so as a $\mathbf{k}[x]$–module*

$$HH_n(\Lambda[x]) = HH_n(\Lambda) \otimes_{\mathbf{k}} \mathbf{k}[x] \;\oplus\; HH_{n-1}(\Lambda) \otimes_{\mathbf{k}} \mathbf{k}[x]dx$$

Proof The bar complex $C_\bullet(\mathbf{k}[x])$ used to compute $HH_*(\mathbf{k}[x])$ is free over $\mathbf{k}$ as is the resulting homology. Thus the result follows from the Künneth Tensor Formula [Mac, p. 166].

By the Künneth Formula, $HH_n(\Lambda[x])$ is generated by cycles of the form $\alpha\#x^i$ and $\beta\#x^{i-1}dx$, where α is a cycle in $C_n(\Lambda)$ and β is a cycle in $C_{n-1}(\Lambda)$. Note that we are identifying dx with $[x]$ (via the isomorphism $HH_1(\mathbf{k}[x]) \approx \Omega_{\mathrm{k}[x]/\mathrm{k}}$), so as to not confuse the latter with the homothety $[r]$.

Theorem 4.4 *Let $\mathbf{k}$ be a commutative ring, R a commutative $\mathbf{k}$–algebra and Λ an R–algebra. Let α be a cycle in $HH_n(\Lambda)$ and β be a cycle in $HH_{n-1}(\Lambda)$. Then in $HH_n(\Lambda[x])$ we have the following formulas for $V_m, [r], F_m$ where $i \geq 1$, $r \in R$.*

$$\begin{aligned}
V_m(\alpha\#x^i) &= \alpha\#x^{im} \\
[r](\alpha\#x^i) &= r^i\alpha\#x^i \\
F_m(\alpha\#x^i) &= \begin{cases} m\alpha\#x^{i/m} & \text{if } m|i \\ 0 & \text{otherwise} \end{cases} \\
&\quad \text{and} \\
V_m(\beta\#x^{i-1}dx) &= \beta\#x^{m(i-1)}dx^m = m\beta\#x^{im-1}dx \\
[r](\beta\#x^{i-1}dx) &= \beta\#(rx)^{i-1}d(rx) = r^i\beta\#x^{i-1}dx + (\beta\#r^{i-1}dr)\#x^i \\
F_m(\beta\#x^{i-1}dx) &= \begin{cases} \beta\#x^{i/m-1}dx & \text{if } m|i \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

The first step in proving the above theorems is the special case $\Lambda = \mathbf{k}$. Recall from [HKR] that there is a map $\Omega^n_{R/\mathrm{k}} \to HH_n(R)$ given by

$$a_0da_1 \wedge da_2 \wedge \cdots \wedge da_n \mapsto \sum_{\sigma\in\Sigma_n} \mathrm{sgn}(\sigma)a_0[a_{\sigma^{-1}(1)}|\cdots|a_{\sigma^{-1}(n)}]$$

where Σ_n is the symmetric group on n elements. If R is smooth over $\mathbf{k}$ or if $n = 0, 1$ then it is classical that this map is an isomorphism. If $\mathbf{k}$ contains $\mathbf{Q}$ then it is also well known that the image is a direct summand of $HH_n(R)$. It is clear that these results extend to the case of $N\Omega^n_{R/\mathrm{k}} \to NHH_n(R)$.

Theorem 4.5 *The VHF operators commute with the map $\phi : N\Omega^n_{R/k} \to NHH_n(R)$. Here the VHF operators are defined as in 4.1 for HH_* and as in 3.1 for $N\Omega^n_R$.*

Proof The result is clear for V_m and $[r]$ since these endomophisms are induced by ring homomorphisms ($x \mapsto x^m$ and $x \mapsto rx$, respectively).

As in §3 every element of $N\Omega_R$ can be written as a sum of differentials $x^i\omega$ and $x^{i-1}\nu \wedge dx$ where $\omega \in \Omega^n_R$ and $\nu \in \Omega^{n-1}_R$. For the first case, $\phi(x^i\omega)$ is a sum of chains of the form $b_0x^i[b_1|\cdots|b_n]$ where the $b_j \in R$. To apply F_m to this we first apply $\imath$ and then the trace. But $\imath(b_j)$ is a diagonal matrix so all non-zero terms of the trace must also take the diagonal of the lefthand factor in formula (1). But $\imath(b_0x^i)$ has diagonal terms only if i is a multiple of m and then each diagonal entry is $b_0x^{i/m}$. Thus $F_m(b_0x^i[b_1|\cdots|b_n]) = mb_0x^{i/m}[b_1|\cdots|b_n]$ from which it follows easily that $F_m(\phi x^i\omega) = \phi(mx^{i/m}\omega) = \phi F_m(x^i\omega)$. The second case is similar in that each term of $\phi(x^{i-1}\nu \wedge dx)$ has a x^{i-1} and x in two positions, and no other powers of x. The nonzero terms in $\imath(x)$ lie just below the diagonal (modulo m), and the terms in $\imath(x^{i-1})$ lying just above the diagonal (modulo m) are zero unless i is a multiple of m, when they are $x^{i/m}$.

Corollary 4.6 *(The case $\Lambda = \mathbf{k}$) The isomorphism $\phi : N\Omega^n_{\mathbf{k}} \approx NHH_n(\mathbf{k})$ commutes with V_m, F_m and $[r]$ for $r \in \mathbf{k}$. The Carf($\mathbf{k}$)–module structures on $x\mathbf{k}[x] = NHH_0(\mathbf{k})$ and $\Omega(\mathbf{k}[x]/\mathbf{k}) = NHH_1(\mathbf{k})$ which were described in 2.9 and 2.10 are induced by the natural VHF operators defined in this section.*

Recall that the shuffle product # is a chain complex map. As such it gives rise to maps $HH_p(\Lambda) \otimes HH_q(R[x]) \to HH_{p+q}(\Lambda[x])$ whenever Λ is an R–algebra and R is a commutative $\mathbf{k}$–algebra [Mac, Chap X, also p 313]. The following is immediate from the definitions of the VHF operators.

Lemma 4.7 *Let Λ be a R–algebra where R is a commutative $\mathbf{k}$–algebra. Let $p = 0, 1$ and $q = 1 - p$. Then the following formulas hold in $NHH_{p+q}(\Lambda)$ for $\alpha \in HH_p(\Lambda)$, $\beta \in NHH_q(R)$, $m > 0$ and $r \in R$.*

$$\begin{aligned} V_m(\alpha\#\beta) &= \alpha\#V_m(\beta) \\ [r](\alpha\#\beta) &= \alpha\#[r](\beta) \\ F_m(\alpha\#\beta) &= \alpha\#F_m(\beta) \end{aligned}$$

At last we can give the proofs of our main theorems!

Proof of Theorem 4.4 The formulas for V_m, F_m follow from 4.7, 4.6, 3.1 and examples 2.9, 2.10. Also using 4.7, we can regard x^i as a cycle in $NHH_0(R) = xR[x]$ and $x^{i-1}dx$ as a cycle in $NHH_1(R) = \Omega_{R[x]/k}$ so $[r](\alpha\#x^i) = \alpha\#[r]x^i = \alpha\#(rx)^i = r^i\alpha\#x^i$

and $V_m(\beta \# x^{i-1}dx) = \beta\#[r](x^{i-1}dx) = \beta\#(r^i x^{i-1}dx + r^{i-1}x^i dr) = \beta\# r^i x^{i-1}dx + \beta\# r^{i-1}dr\# x^i$ by 3.1.

Proof of Theorem 4.2 We shall show that the Cartier identities of Theorem 2.8 hold. Since by 4.4, $HH_n(\Lambda)\otimes x\mathbf{k}[x]$ is closed under the action of V_m, [r] and F_m, it will then be a *Carf(R)*–submodule. Now V_m and F_m respect the direct sum decomposition of the Künneth Formula, so identities (i-V,F), (ii-V,F), (iv), (v) and (vii) follow from the special case $\Lambda = \mathbf{k}$ and the results of Sections 2 and 3. Identities (i-[r]), (ii-[r]) and (iii-V) follow immediately from the definitions. Thus it is necessary only to demonstrate axioms (iii-F) and (vi). But again we use the formulas of Theorem 4.4 and the argument then follows closely that of Theorem 3.2.

Since $W(R)$ is contained in *Carf(R)* the above theorems immediately yield:

Corollary 4.8 *With notation as in 4.4, $NHH_n(\Lambda)$ is a $W(R)$–module with operations:*

$$\begin{aligned}(1-rt^m)*(\alpha\# x^i) &= mr^{i/m}\# x^i\\ (1-rt^m)*(\beta\# x^{i-1}dx) &= mr^{i/m}\beta\# x^{i-1}dx + (\beta\# r^{i/m-1}dr)\# x^i\end{aligned}$$

if $m|i$ and 0 otherwise.

As an application, we can generalize the module structure of Corollary 3.6 to non-commutative de Rham cohomology. There are well known maps (see e.g. [LQ]) $B : HC_n(\Lambda) \to HH_{n+1}(\Lambda)$ and $I : HH_n(\Lambda) \to HC_n(\Lambda)$, which when composed give a map $BI : HH_n(\Lambda) \to HH_{n+1}(\Lambda)$. For example, the cokernel of $BI : HH_0(\Lambda) \to HH_1(\Lambda)$ is $HC_1(\Lambda)$. Since $(BI)^2 = 0$ we get a complex

$$0 \longrightarrow HH_0(\Lambda) \xrightarrow{BI} HH_1(\Lambda) \xrightarrow{BI} HH_2(\Lambda) \xrightarrow{BI} \cdots$$

The cohomology of this complex is called the non-commutative de Rahm cohomology of Λ (see eg. [LQ]). We will use the notation $HH^*_{dR}(\Lambda)$ to distinguish it from the classical de Rahm cohomology $H^*_{dR}(\Lambda)$ used in §3. Of course, if Λ is commutative and smooth over $\mathbf{k}$ then H^*_{dR} and HH^*_{dR} agree. Finally, we use the notation $NHH^n_{dR}(\Lambda) = \ker(HH^n_{dR}(\Lambda[x]) \xrightarrow{x\mapsto 0} HH^n_{dR}(\Lambda))$. This is known to be a torsion abelian group. In particular, if $\mathbf{Q} \subseteq \mathbf{k}$ then $NHH^*_{dR}(\Lambda) = 0$.

Proposition 4.9 *Let R be a commutative $\mathbf{k}$–algebra and suppose that Λ is a (possibly non-commutative) R–algebra. The differential $BI : NHH_n(\Lambda) \to NHH_{n+1}(\Lambda)$ is a morphism of Carf(R)–modules. In particular,*

- *$NHH^n_{dR}(\Lambda)$ is a Carf(R)–module.*
- *$NHC_1(\Lambda)$ is a Carf(R)–module*

Proof: The VHF endomorphisms are endomorphisms of mixed complexes. As such they commute with S, B, I and the composition BI. Hence BI commutes with every element of *Carf(R)*.

We can now show that $NHC_*(\Lambda)$ is also a *Carf(R)*–module, at least if we assume that $\mathbf{Q} \subseteq \mathbf{k}$. It follows from [Goodw] that the SBI sequence breaks up into short exact sequences

$$0 \longrightarrow NHC_{n-1}(\Lambda) \xrightarrow{B} NHH_n(\Lambda) \xrightarrow{I} NHC_n(\Lambda) \longrightarrow 0$$

¿From this we see that $NHC_{n-1}(\Lambda)$ is the kernel of $NHH_n(\Lambda) \xrightarrow{BI} NHH_{n+1}(\Lambda)$. The following follows immediately from 4.9.

Theorem 4.10 *Let $\mathbf{Q} \subseteq \mathbf{k}$ and R, Λ as above. Then each $NHC_n(\Lambda)$ is a Carf(R)–module, and*

$$0 \longrightarrow NHC_{n-1}(\Lambda) \xrightarrow{B} NHH_n(\Lambda) \xrightarrow{I} NHC_n(\Lambda) \longrightarrow 0$$

is a short exact sequence of Carf(R)–modules.

As an illustration of the power of module structures, we give a quick proof of a calculation of Kassel [K-CCM, Example 3.3]. For this, we need the Leibniz formula [LQ, p.576], [FT] (see also [GRW, 5.5]):

$$BI(\alpha\#\beta) = BI(\alpha)\#\beta + (-1)^p\alpha\#BI(\beta),$$

where $\alpha \in HH_p(\Lambda)$, and $\beta \in HH_q(\mathbf{k}[x])$. Identifying $HH_1(\mathbf{k}[x]) = \Omega_{\mathbf{k}[x]}$, we also note that $BI(dx) = BI(BI(x)) = 0$.

Proposition 4.11 *Let $\mathbf{k}$ be any commutative ring containing $\mathbf{Q}$, and let Λ be any (not necessarily commutative) $\mathbf{k}$–algebra. Then there is a natural isomorphism of Carf($\mathbf{k}$)–modules:*

$$NHC_n(\Lambda) \approx HH_n(\Lambda) \otimes x\mathbf{k}[x].$$

The module structure on the right is given in 2.9. Consequently,

$$HC_n(\Lambda[x]) = HC_n(\Lambda) \oplus HH_n(\Lambda) \otimes x\mathbf{k}[x].$$

Proof By Theorem 4.10 above NHC_n is a *Carf*($\mathbf{k}$)–module, so by Corollary 2.14 it is enough to show that for some $r \in \mathbf{k}$, $r \neq 0, \pm 1$, the eigenspace M_1 of r under the endomorphism $(1 - rt)*$ is isomorphic to $HH_n(\Lambda)$. Viewing $NHC_n(\Lambda)$ as the image of $BI : NHH_n(\Lambda) \to NHH_{n+1}(\Lambda)$, BI is an epimorphism of *Carf*($\mathbf{k}$)–modules so M_1 is the image of the corresponding eigenspace of $NHH_n(\Lambda)$ which is seen to be generated by $\alpha\#x$, and $\beta\#dx$, $\alpha \in HH_n(\Lambda)$ and $\beta \in HH_{n-1}(\Lambda)$ by Theorem 4.4.

Now define $\Psi : HH_n(\Lambda) \to M_1$ by $\Psi(\alpha) = BI(\alpha\#x) = BI(\alpha)\#x + (-1)^n\alpha\#dx$. But $BI(\beta\#dx) = BI(\beta)\#dx = (-1)^n BI(BI(\beta)\#x) = (-1)^n\Psi(BI(\beta))$ so Ψ is surjective. Projecting onto the summand $HH_n(\Lambda) \otimes \mathbf{k}[x]dx$ of $NHH_{n+1}(\Lambda)$ (via the Künneth Formula) we see that Ψ is injective, completing the proof.

To conclude this section, we observe that the VHF operators commute with the Dennis Trace map $K_*(\Lambda[x]) \to HH_*(\Lambda[x])$. [Igusa, 1.a]. This is because the Verschiebung $NK_n(\Lambda) \to NK_n(\Lambda)$ is induced by the map $x \to x^m$, the homothety $[r]$ is induced by $x \mapsto rx$, and the Frobenius is the transfer on K-theory associated with this map (see [Bloch2, W-MVS]). We note that the Frobenius transfer (associated with $x \mapsto x^m$) is given by composing the map on K-theory induced by $\imath$ with the isomorphism $K_n(M_m(\Lambda)) \approx K_n(\Lambda)$ induced by Morita invariance. Since these operations make $NK_n(\Lambda)$ into a *Carf(R)*–module [W-MVS], we have the following result.

Proposition 4.12 *Let Λ be a R algebra where R is a commutative $\mathbf{k}$–algebra. Then the Dennis Trace map $D : NK_n(\Lambda) \to NHH_n(\Lambda)$ is a homomorphism of Carf(R)–modules.*

5 The $W(R)$–module structure on HH_* of graded rings

In this section $\mathbf{k}$ will be a commutative ring, R a commutative $\mathbf{k}$–algebra and $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ will be a graded unitary R–algebra, i.e. each graded piece Λ_j is a R–module.

Set $\widetilde{HH}_n(\Lambda) = HH_n(\Lambda)/HH_n(\Lambda_0)$. We will show that $\widetilde{HH}_n(\Lambda)$ is a $W(R)$–module. The example in [GRW, Thm. 3.11] shows that $\widetilde{HH}_n(\Lambda)$ need not be a *Carf(R)*–module, because Theorem 2.13 is violated for $n \geq 2$.

Given a graded R–algebra Λ and an indeterminant x, we define a map $f_x : \Lambda \to \Lambda[x]$ by sending $a_k \in \Lambda_k$ to $a_k x^k$. This induces a map $HH_n(\Lambda) \to HH_n(\Lambda[x])$ which we also call f_x. If α is an cycle of $HH_n(\Lambda)$ of weight k (see [GRW, Def. 1.1]), using the decomposition of the Künneth Formula we see $f_x(\alpha) = \alpha'\#x^k + \beta\#x^{k-1}dx$ for some $\alpha' \in HH_n(\Lambda)$ and some $\beta \in HH_{n-1}(\Lambda)$. Now $x \mapsto 1$ is a left inverse to f_x so (since $d(1) = 0$) $\alpha' = \alpha$. Writing $\xi_x(\alpha) = \beta$ we have a homomorphism $\xi : HH_n(\Lambda) \to HH_{n-1}(\Lambda)$. We remark that ξ is very dependent on the grading of Λ.

For example, if Λ is the polynomial ring $\Lambda_0[t]$, and t is in degree 1, the map f_x is the substitution $t \mapsto xt$. It is easy to calculate directly that for $\omega \in HH_*(\Lambda_0)$

$$\xi_x(t^k\omega) = 0, \text{ but } \xi_x(\omega\#t^{k-1}dt) = t^k\omega.$$

Lemma 5.1 *With notation as above, the map $\xi : HH_n(\Lambda) \to HH_{n-1}(\Lambda)$ defined by*

$$f_x(\alpha) = \alpha\#x^k + \xi(\alpha)\#x^{k-1}dx$$

satisfies the equation $\xi^2 = 0$.

Proof We consider $\Lambda[y]$ as a graded ring with Λ graded in the usual way and y in degree 0. Consider the following commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{f_y} & \Lambda[y] \\ f_x \downarrow & & \downarrow f_x \\ \Lambda[x] & \xrightarrow{x \mapsto xy} & \Lambda[x,y]. \end{array}$$

Calculating (writing h for the bottom map) we have for ω a homogeneous form of degree k in $HH_n(\Lambda)$

$$\begin{aligned} hf_x(\omega) &= h(\omega\#x^k + \xi(\omega)\#x^{k-1}dx) \\ &= \omega\#(xy)^k + \xi(\omega)\#(xy)^{k-1}d(xy) \\ &= \omega\#x^ky^k + \xi(\omega)\#x^ky^{k-1}dy + \xi(\omega)\#x^{k-1}y^kdx \end{aligned} \tag{6}$$

On the other hand

$$\begin{aligned} f_xf_y(\omega) &= f_x(\omega\#y^k + \xi(\omega)\#y^{k-1}dy) \\ &= \omega\#y^kx^k + \xi(\omega)\#x^ky^{k-1}dy + \xi(\omega\#y^k + \xi(\omega)\#y^{k-1}dy)\#x^{k-1}dx \\ &= \omega\#y^kx^k + \xi(\omega)\#x^ky^{k-1}dy + \xi(\omega\#y^n)\#x^{k-1}dx \\ &\qquad + \xi(\xi(\omega)\#y^{k-1}dy)\#x^{k-1}dx \end{aligned} \tag{7}$$

Now to calculate the last two terms in (7) we have

$$\begin{aligned} f_x(\omega\#y^k) &= f_x(\omega)\#f_x(y^k) \\ &= (\omega\#x^k + \xi(\omega)\#x^{k-1}dx)\#y^k = \omega\#x^ky^k + \xi(\omega)\#x^{k-1}y^kdx \end{aligned} \tag{8}$$

since y has degree 0, so that $\xi(\omega\#y^k) = \xi(\omega)\ y^k$. Also

$$\begin{aligned} f_x(\xi(\omega)\#y^{k-1}dy) &= f_x(\xi(\omega))\#f_x(y^{k-1}dy) \\ &= (\xi(\omega)x^k + \xi(\xi(\omega))x^{k-1}\#dx)\#y^{k-1}dy \\ &= \xi(\omega)x^ky^{k-1}dy + \xi^2(\omega)\#x^{k-1}y^{k-1}dx \wedge dy \end{aligned} \tag{9}$$

so that $\xi(\xi(\omega)\#y^{k-1}dy) = \xi^2(\omega)\#y^{k-1}dy$. Substituting (8) and (9) into (7) and comparing with (6) we see that $\xi^2(\omega)\#y^{k-1}x^{k-1}dy \wedge dx = 0$. But this term lives in the last summand of the Künneth Formula decomposition

$$HH_n(\Lambda[x,y]) = HH_n(\Lambda) \otimes_{\mathrm{k}} \mathbf{k}[x,y] \oplus HH_{n-1}(\Lambda) \otimes_{\mathrm{k}} \Omega_{\mathbf{k}[x,y]} \oplus HH_{n-2}(\Lambda) \otimes_{\mathrm{k}} \Omega^2_{\mathbf{k}[x,y]}.$$

Since $\Omega^2_{\mathbf{k}[x,y]} = \mathbf{k}[x,y]dx \wedge dy$ is a free $\mathbf{k}$–module it follows that $\xi^2(\omega) = 0$. This completes the proof of the lemma.

We can now prove our main theorem of this section.

Theorem 5.2 *Let Λ be a graded R–algebra where R is a commutative $\mathbf{k}$–algebra. Then there is a $W(R)$–module structure on $\widetilde{HH}_n(\Lambda)$ such that the inclusion $f_x : \widetilde{HH}_n(\Lambda) \to NHH_n(\Lambda)$ is a $W(R)$–module map. In particular, if ω is a homogeneous cycle of weight k then*

$$(1 - rt^m) * \omega = mr^{k/m}\omega + r^{k/m-1}\xi(\omega)\#dr \tag{10}$$

if m divides k and 0 otherwise.

Proof We will show the following diagram commutes

$$\begin{array}{ccc} \widetilde{HH}_n(\Lambda) & \xrightarrow{f_x} & NHH_n(\Lambda) \\ \Big\downarrow * & & \Big\downarrow * \\ \widetilde{HH}_n(\Lambda) & \xrightarrow{f_x} & NHH_n(\Lambda) \end{array}$$

where $*$ denotes multiplication by $(1 - rt^m)$ given by formula (10) on the left and given by Corollary 4.8 on the right. Then, since $NHH_n(\Lambda)$ is a $W(R)$–module by 4.8 and f_x is injective it follows that $\widetilde{HH}_n(\Lambda)$ is a $W(R)$–module.

To this end, let ω be a homogeneous cycle in $\widetilde{HH}_n$ of weight k. If m does not divide k then both multiplications by $(1 - rt^m)$ give 0. If $m|k$ then as in the proof of Lemma 5.1 we calculate

$$\begin{aligned} \xi(mr^{k/m}\omega + r^{k/m-1}\xi(\omega)\#dr) &= \xi(mr^{k/m}\omega) + \xi(r^{k/m-1}\xi(\omega)\#dr) \\ &= mr^{k/m}\xi(\omega) + r^{k/m-1}\xi^2(\omega)\#dr = mr^{k/m}\xi(\omega) \end{aligned}$$

using the fact that r is of degree 0 and $\xi^2(\omega) = 0$. From this the easy calculation

$$\begin{aligned} (1 - rt^m) * f_x(\omega) &= mr^{k/m}\omega\#x^k + mr^{k/m}\xi(\omega)\#x^{k-1}dx + r^{k/m-1}\xi(\omega)\#x^k dr \\ &= f_x(1 - rt^m) * \omega \end{aligned}$$

completes the proof.

Remark 5.3 If Λ is commutative the map $\Omega^n_{\Lambda/\mathrm{k}} \to HH_n(\Lambda)$ induces a map (which we call ϕ as in 4.5) $\tilde{\Omega}^n_{\Lambda/\mathrm{k}} \to \widetilde{HH}_n(\Lambda)$. This map is a map of $W(R)$–modules. In fact, we can mimic the definition of ξ in the context of $\tilde{\Omega}^n_{\Lambda/\mathrm{k}}$. For if $\omega = a_0 da_1 \wedge da_2 \wedge \cdots \wedge da_n$ where $e_i = \deg(a_i)$ and $k = e_0 + \cdots + e_n$ then

$$\begin{aligned} f_x(\omega) &= a_0 x^{e_0} d(a_1 x^{e_1}) \wedge \cdots \wedge d(a_n x^{e_n}) \\ &= x^k \omega + a_0 x^{k-e_i} \sum_{i=1}^{n} a_i da_1 \wedge \cdots \wedge d(x^{e_i}) \wedge \cdots \wedge da_n \\ &= x^k \omega + a_0 x^{k-1} \sum_{i=1}^{n} e_i a_i da_1 \wedge \cdots \wedge dx \wedge \cdots \wedge da_n \\ &= x^k \omega + a_0 x^{k-1} \sum_{i=1}^{n} (-1)^{n-i} e_i a_i da_1 \wedge \cdots \wedge da_{i-1} \wedge da_{i+1} \wedge \cdots \wedge da_n \wedge dx \end{aligned}$$

Thus we may write

$$\xi(\omega) = a_0 \sum_{i=1}^{n} (-1)^{n-i} e^i a_i da_1 \wedge \cdots \wedge \widehat{da_i} \wedge \cdots \wedge da_n$$

Hence formula (1) of Theorem 3.4 could have been written

$$(1 - rt^m) * \omega = r^{k/m} m\omega + r^{k/m-1} \xi(\omega) \wedge dr$$

which mirrors Theorem 5.2. In particular, if Λ is smooth over k then the formulas in Theorems 3.4 and 5.2 agree.

Our technique for obtaining the $W(R)$–module structure on $\widetilde{HH}_*(\Lambda)$, i.e. imbedding $\widetilde{HH}_*(\Lambda)$ in $NHH_*(\Lambda)$, is analagous to the method used in [W-MGR, §1] and thus we obtain

Theorem 5.4 *With* $\mathbf{k}$, R, *and* Λ *as in Theorem 5.2 the Dennis trace map* D : $\tilde{K}_n(\Lambda) \to \widetilde{HH}_n(\Lambda)$ *is a* $W(R)$*–module homomorphism.*

When Λ is a graded $\mathbf{k}$–algebra, the map BI induces $BI : \widetilde{HH}_n(\Lambda) \to \widetilde{HH}_{n+1}(\Lambda)$. The homology of the resulting complex will be denoted $\widetilde{HH}^*_{dR}(\Lambda)$. Alternatively we could define $\widetilde{HH}^*_{dR}(\Lambda)$ to be the quotient $HH^*_{dR}(\Lambda)/HH^*_{dR}(\Lambda_0)$, as in §3 and §4. We then have

Theorem 5.5 *Let R be a commutative* **k** *algebra and Λ an graded R–algebra. Then $BI : \widetilde{HH}_n(\Lambda) \to \widetilde{HH}_{n+1}(\Lambda)$ is a $W(R)$–module map. Consequently, $\widetilde{HH}^n_{dR}(\Lambda)$ is a $W(R)$–module for each n.*

Proof Since B and I are natural and hence commute with the embedding f_x of $\widetilde{HH}_n(\Lambda)$ in $NHH_n(\Lambda)$, this follows from the corresponding result for NHH. This was proven in 4.9.

Finally we derive the graded analogues of the results in the last section about NHC when $\mathbf{Q} \subseteq \mathbf{k}$. If Λ is a graded $\mathbf{k}$–algebra, $\widetilde{HC}_n(\Lambda)$ will denote the quotient $HC_n(\Lambda)/HC_n(\Lambda_0)$. As in loc. cit., it follows from [Goodw] that the SBI sequence breaks up into short exact sequences

$$0 \longrightarrow \widetilde{HC}_{n-1}(\Lambda) \xrightarrow{B} \widetilde{HH}_n(\Lambda) \xrightarrow{I} \widetilde{HC}_n(\Lambda) \longrightarrow 0.$$

¿From this we see that $\widetilde{HC}_{n-1}(\Lambda)$ is the kernel of the map $\widetilde{HH}_n(\Lambda) \xrightarrow{BI} \widetilde{HH}_{n+1}(\Lambda)$). Since BI is a $W(R)$–module map we have

Corollary 5.6 *Let $\mathbf{Q} \subseteq \mathbf{k}$ and R, Λ as above, with Λ a graded R–algebra. Then $\widetilde{HC}_n(\Lambda) = HC_n(\Lambda)/HC_n(\Lambda_0)$ is a $W(R)$–module.*

6 The $W(\mathbf{k})$ and *Carf*($\mathbf{k}$)–module structure on HC_* in positive characteristic

In this section, we give a mixed complex $C_\bullet(\Lambda) \otimes N\Omega^\bullet_{\mathbf{k}}$ of *Carf*($\mathbf{k}$)–modules whose Hochschild and cyclic homology are $NHH(\Lambda)$ and $NHC(\Lambda)$. Under the induced *Carf*($\mathbf{k}$)–module structure on $NHH(\Lambda)$ and $NHC(\Lambda)$, we show that V_m, F_m and $[r]$ with $r \in \mathbf{k}$ correspond to the VHF operators on $C_\bullet(\Lambda[x])$. This will establish the following result:

Theorem 6.1 *Let Λ be a* **k**–*algebra. The VHF operators on $C_\bullet(\Lambda[x])$ defined in §4 make $NHH_*(\Lambda)$ and $NHC_*(\Lambda)$ into Carf(*$\mathbf{k}$*)–modules.*

Recall from [K-CCM] that a *mixed complex* (M, b, B) of **k**–modules is a graded **k**–module $\{M_i : i \geq 0\}$, together with endomorphisms b and B of degree -1 and $+1$ respectively, which satisfy

$$b^2 = B^2 = Bb + bB = 0$$

Associated to a mixed complex are its Hochschild homology $HH_*(M) = H_*(M, b)$, its cyclic homology $HC_*(M) = H_*({}_BM, d)$ (see [K-CCM, p. 198]) and related groups such as $HP_*(M)$ etc.

Example 6.2 If R is a commutative $\mathbf{k}$–algebra, $(\Omega^\bullet_R, 0, d)$ is a mixed complex of R–modules, where $d : \Omega^n_R \to \Omega^{n+1}_R$ is the de Rahm differential. By Theorem 3.2 the mixed subcomplex $(N\Omega^\bullet_R, 0, d)$ of $(\Omega^\bullet_{R[x]}, 0, d)$ is even a mixed complex of $\mathbf{k}\otimes$ *Carf(R)*–modules.

Example 6.3 If Λ is any $\mathbf{k}$–algebra, the "reduced bar construction" (see [CE, p. 176] or [Mac, X.2]) is the complex $C_\bullet(\Lambda)$ with

$$C_n(\Lambda) = \Lambda \otimes_{\mathrm{k}} (\overbrace{\Lambda/\mathbf{k} \otimes \cdots \otimes \Lambda/\mathbf{k}}^{n \text{ times}})$$

This is a mixed complex, but not a cyclic module in the sense of Connes. It is traditional to use the notation $a_0[a_1|a_2|\cdots|a_n]$ for a generator of $C_n(\Lambda)$. The endomorphism B is the usual one given in [LQ, K-CCM] etc.

Example 6.4 The tensor product of two mixed complexes of $\mathbf{k}$–modules is determined in the usual way. In particular, we can combine the above two examples to construct $(C_\bullet(\Lambda) \otimes_{\mathrm{k}} N\Omega^\bullet_{\mathrm{k}},\ b \otimes 1,\ B \otimes 1 \pm 1 \otimes d)$, which is not only a mixed complex of $\mathbf{k}$–modules, but is also a mixed complex of *Carf(*$\mathbf{k}$*)*–modules.

In order to show that HH and HC of the previous mixed complex agree with $NHH(\Lambda)$ and $NHC(\Lambda)$, we need to compare it to the mixed complex $NC_\bullet$. For this, we need to recall the homotopy notion of morphism for mixed complexes. There are two notions of morphisms of mixed complexes: *strict morphisms* (map commuting with b and B) and *strongly homotopy maps* [K-CCM, 2.2]. If M and N are mixed complexes a *strongly homotopy map* $G : M \to N$ is a collection of graded $\mathbf{k}$–module maps $G^{(i)} : M_\bullet \to N_{\bullet+2i}, i \geq 0$ such that $G^{(0)}$ is a chain map ($[G^{(0)}, b] = 0$) and

$$[G^{(i+1)}, b] + [G^{(i)}, B] = 0 \text{ for all } i \geq 0$$

(In [K-CCM] the brackets stand for graded commutators but as the $G^{(i)}$ have even degree here they are simply commutators.) The composition $H \circ G$ of strongly homotopy maps is again a strongly homotopy map: $(HG)^{(n)} = \Sigma H^{(i)} G^{(j)}$.

By [K-CCM, 2.3] a strongly homotopy map not only induces the obvious map $G_* : HH_*(M) \to HH_*(N)$ but also maps $G_* : HC_*(M) \to HC_*(N)$, etc.

We will call a map $G : M \to N$ (strict or strongly homotopy) a *quasi-isomorphism* if the maps $HH_*(M) \to HH_*(N)$ induced by $G^{(0)}$ are isomorphisms. It is well known that the induced maps $G_* : HC_*(M) \to HC_*(N)$ etc. are also isomorphisms whenever G is a quasi-isomorphism of mixed complexes.

Example 6.5 (homothety) If $r \in \mathbf{k}$, the map $R[x] \to R[x]$ sending x to rx defines a strict endomorphism $[r]$ of the mixed complex $N\Omega^\bullet_R$ of Example 6.2. If we want to define a homothety $[r]$ for other elements r of R, we need a slightly different construction, owing to the fact that dr need not vanish.

Consider the group $\Omega^\bullet_R$ as a functor of R and apply this to the map $\mathbf{k}[x] \to \mathbf{k}[x,r]$ sending $x \mapsto rx$, where r is an indeterminate. We let $[r]$ denote the resulting map $\Omega^\bullet_{\mathrm{k}[x]} \to \Omega^\bullet_{\mathrm{k}[x,r]} = \Omega^\bullet_{\mathrm{k}[r]} \otimes \Omega^\bullet_{\mathrm{k}[x]}$. In particular we have

$$\begin{aligned} [r](x^i) &= r^i \otimes x^i \\ [r](x^{i-1}dx) &= (rx)^{i-1}d(rx) = r^i \otimes x^{i-1}dx + r^{i-1}dr \otimes x^i \end{aligned}$$

A simple calculation (left to the reader) shows that $[r]$ is a strict morphism of the mixed complexes of Example 6.2 (Cf. 3.1). Composing with $\mathbf{k}[r] \to R$ yields another strict morphism $[r] : \Omega^\bullet_{\mathrm{k}[x]} \to \Omega^\bullet_R \otimes \Omega^\bullet_{\mathrm{k}[x]}$.

Example 6.6 By [Rine], [HK] or [K-CCM, 2.4] there is a strongly homotopy map $\nabla : C_\bullet(\Lambda) \otimes C_\bullet(\Lambda') \to C_\bullet(\Lambda \otimes \Lambda')$ of mixed complexes, natural in Λ and Λ', such that $\nabla^{(0)}$ is the shuffle map, $\nabla^{(1)}$ is Rinehart's cyclic shuffle map and $\nabla^{(i)} = 0$ for $i > 1$. Since the shuffle map is a quasi-isomorphism, so is ∇.

Example 6.7 For the polynomial ring $\mathbf{k}[x]$ there is a strict morphism of mixed complexes $\eta : C_\bullet(\mathbf{k}[x]) \to \Omega^\bullet_{\mathrm{k}[x]}$ with η_0 the canonical isomorphism $C_0(\mathbf{k}[x]) \approx \mathbf{k}[x] = \Omega^0_{\mathrm{k}[x]}$ and $\eta_1(f \otimes g) = f\,dg$. Since it induces an isomorphism on HH_*, it is a quasi-isomorphism. There is no strict morphism going the other way, however; one needs a strongly homotopy map for that.

Lemma 6.8 *The chain map $g_j : \Omega^j_{\mathrm{k}[x]} \to C_j(\mathbf{k}[x])$ defined by the formulas $g_0(x^i) = x^i$, $g_1(x^{i-1}dx) = x^{i-1}[x]$ extends to a strongly homotopy map*

$$G : \Omega^\bullet_{\mathrm{k}[x]} \to C_\bullet(\mathbf{k}[x])$$

such that ηG is the identity on $\Omega^\bullet_{\mathrm{k}[x]}$.

Proof Because $b : C_1(\mathbf{k}[x]) \to C_0(\mathbf{k}[x])$ is zero, $g = G^{(0)}$ is a chain map. Because $G^{(i+1)}b = 0$, ($b = 0$ on $\Omega^\bullet_{\mathrm{k}[x]}$) we need $G^{(j)}_0 : \mathbf{k}[x] \to C_{2j}(\mathbf{k}[x])$ and $G^{(j)}_1 : \Omega^1_{\mathrm{k}[x]} \to C_{2j+1}(\mathbf{k}[x])$ such that

$$bG^{(j+1)}_0 = G^{(j)}_1 B - BG^{(j)}_0 \quad \text{and} \quad bG^{(j+1)}_1 = -BG^{(j)}_1 \tag{11}$$

For $G^{(1)}_0$ we noted in §4 that the map $\Omega^1_{\mathrm{k}[x]} \to C_1(\mathbf{k}[x]) \to HH_1(\mathbf{k}[x])$ given by $x^i\,dx^j \mapsto x^i[x^j]$ is an isomorphism. Thus it follows that the cycles $G^{(0)}_1 B(x^i) = ix^{i-1}[x]$

and $BG_0^{(j)}(x^i) = [x^i]$ map to the same element in $HH_1(\mathbf{k}[x])$ and hence the image of $G_1^{(j)}B - BG_0^{(j)}$ is contained in the image of b, thus it is possible to define $G_0^{(1)}$ satisfying the first identity of Equation (11).

We now argue by induction and assume that $G^{(j)}$ is defined and satisfies Equation (11). Then $b(G_1^{(j)}B - BG_0^{(j)}) = bG_1^{(j)}B - bBG_0^{(j)} = bG_1^{(j)}B + BbG_0^{(j)} = -BG_1^{(j-1)}B + B(G_1^{(j-1)}B - BG_0^{(j-1)}) = 0$. Now since the complex $C_\bullet(\mathbf{k}[x])$ is exact for $\bullet \geq 2$ we can define $G_0^{(j+1)}$ to satisfy Equation (11). Likewise $bBG_1^{(j)} = -BbG_1^{(j)} = BBG_1^{(j-1)} = 0$ so we can likewise define $G_1^{(j+1)}$.

Remark 6.9 Because g is a chain map, ηg is the identity on $\Omega^\bullet_{\mathrm{k}[x]}$ and $g\eta$ is chain homotopic to the identity on $C_\bullet(\mathbf{k}[x])$ by a homotopy ϕ, $(\Omega^\bullet_{\mathrm{k}[x]}, C_\bullet(\mathbf{k}[x]), g, \eta, \phi)$ is a "deformation retract" in the sense of [K-LP]. Since η is a strict morphism, Lemma 6.8 is seen to be an application of [K-LP, Cor. 7.2].

Theorem 6.10 *For every* $\mathbf{k}$*–algebra* Λ*, there are natural quasi-isomorphisms of mixed complexes of* $\mathbf{k}$*-modules*

$$\begin{array}{ccccc} C_\bullet(\Lambda)\otimes\Omega^\bullet_{\mathrm{k}[x]} & \stackrel{1\otimes\eta}{\longleftarrow} & C_\bullet(\Lambda)\otimes C_\bullet(\mathbf{k}[x]) & \stackrel{\nabla}{\longrightarrow} & C_\bullet(\Lambda[x]) \\ C_\bullet(\Lambda)\otimes N\Omega^\bullet_{\mathrm{k}} & \stackrel{1\otimes\eta}{\longleftarrow} & C_\bullet(\Lambda)\otimes NC_\bullet(\mathbf{k}) & \stackrel{\nabla}{\longrightarrow} & NC_\bullet(\Lambda) \end{array}$$

Since $C_\bullet(\Lambda)\otimes N\Omega^\bullet_{\mathbf{k}}$ *is a mixed complex of Carf(*$\mathbf{k}$*)–modules by Theorem 3.2, it follows that there is a Carf(*$\mathbf{k}$*)–module structure on* $NHH_*(\Lambda), NHC_*(\Lambda)$*, etc.*

Proof This follows from 6.6, 6.8 and the Künneth formula.

To finish the proof of Theorem 6.1, we need only show that this structure comes from the VHF operators of §4, which were defined on the mixed complex $C_\bullet(\Lambda[x])$.

Lemma 6.11 *The Verschiebung endomorphisms* V_m *on* $NHH_*(\Lambda)$*, and* $NHC_*(\Lambda)$ *are induced by the strict map* $1\otimes V_m$ *on* $C_\bullet(\Lambda)\otimes N\Omega^\bullet_{\mathbf{k}}$

Proof If $y = x^m$, naturality of ∇ and η, 6.6 and 6.8 yield a commutative diagram of mixed complexes and strongly homotopy maps:

$$\begin{array}{ccccc} C_\bullet(\Lambda)\otimes\Omega^\bullet_{\mathrm{k}[y]} & \stackrel{1\otimes\eta}{\longleftarrow} & C_\bullet(\Lambda)\otimes C_\bullet(\mathbf{k}[y]) & \stackrel{\nabla}{\longrightarrow} & C_\bullet(\Lambda[y]) \\ {\scriptstyle 1\otimes V_m}\downarrow & & \downarrow{\scriptstyle 1\otimes V_m} & & \downarrow{\scriptstyle V_m} \\ C_\bullet(\Lambda)\otimes\Omega^\bullet_{\mathrm{k}[x]} & \stackrel{1\otimes\eta}{\longleftarrow} & C_\bullet(\Lambda)\otimes C_\bullet(\mathbf{k}[x]) & \stackrel{\nabla}{\longrightarrow} & C_\bullet(\Lambda[x]) \end{array}$$

Therefore the vertical maps all induce the same maps on NHH_*, NHC_* etc.

Lemma 6.12 *If $r \in \mathbf{k}$ the homothety $[r]$ on $NHH_*(\Lambda)$ and $NHC_*(\Lambda)$ are induced by the strict map $1 \otimes [r]$ on $C_\bullet(\Lambda) \otimes N\Omega^\bullet_{\mathbf{k}}$.*

Proof Consider the following diagram of mixed complexes of $\mathbf{k}$–modules.

$$\begin{array}{ccccc}
C_\bullet(\Lambda) \otimes \Omega^\bullet_{k[y]} & \xleftarrow{1\otimes\eta} & C_\bullet(\Lambda) \otimes C_\bullet(\mathbf{k}[y]) & \xrightarrow{\nabla} & C_\bullet(\Lambda[y]) \\
{\scriptstyle 1\otimes[r]}\downarrow & & \downarrow{\scriptstyle 1\otimes[r]} & & \downarrow{\scriptstyle [r]} \\
C_\bullet(\Lambda) \otimes \Omega^\bullet_{k[x]} & \xleftarrow{1\otimes\eta} & C_\bullet(\Lambda) \otimes C_\bullet(\mathbf{k}[x]) & \xrightarrow{\nabla} & C_\bullet(\Lambda[x])
\end{array}$$

The vertical maps $[r]$ are the strict morphisms of 4.1 and 3.1. The left square commutes by direct inspection. The right square commutes by naturality of Rinehart's strongly homotopy map ∇.

Lemma 6.13 *The Frobenius endomorphism F_m on $NHH_*(\Lambda)$ and $NHC_*(\Lambda)$ is induced by the strict map $1 \otimes F_m$ on $C_\bullet(\Lambda) \otimes N\Omega^\bullet_{\mathbf{k}}$.*

Proof Consider the following diagram of chain complexes:

$$\begin{array}{ccccc}
C_\bullet(\Lambda) \otimes \Omega^\bullet_{k[x]} & \xleftarrow{1\otimes\eta} & C_\bullet(\Lambda) \otimes C_\bullet(\mathbf{k}[x]) & \xrightarrow{\nabla} & C_\bullet(\Lambda[x]) \\
\Big| & & \downarrow{\scriptstyle 1\otimes\iota} & & \downarrow{\scriptstyle \iota} \\
{\scriptstyle 1\otimes F_m} & & C_\bullet(\Lambda) \otimes C_\bullet(M_m(\mathbf{k}[y])) & \xrightarrow{\nabla} & C_\bullet(M_m(\Lambda[y])) \\
\downarrow & & \downarrow{\scriptstyle trace} & & \downarrow{\scriptstyle trace} \\
C_\bullet(\Lambda) \otimes \Omega^\bullet_{k[y]} & \xleftarrow{1\otimes\eta} & C_\bullet(\Lambda) \otimes C_\bullet(\mathbf{k}[y]) & \xrightarrow{\nabla} & C_\bullet(\Lambda[y])
\end{array}$$

The left hand square commutes by a brute force calculation and the upper right by the naturality of ∇. The lower right square commutes by the explicit definition of ∇. (We learned this calculation from Kassel; see [K-CCB, II.5.4] and [HK].)

These last three lemmas complete the proof of Theorem 6.1.

References

[AT] M. Atiyah and D. Tall, Group representations, λ–rings and the J–homomorphism, Topology 8(1969), 253–297.

[Bloch] S. Bloch, Algebraic K-theory and Crystalline Cohomology, Publ. Math. I.H.E.S. 47(1978), 188–268.

[Bloch2] S. Bloch, Some formulas pertaining to the K-theory of commutative groupschemes, J. Alg 53(1978), 304–326.

[Bour] N. Bourbaki, Algèbra Commutative, Ch. 9, Masson, 1983.

[CE] H. Cartan and S. Eilenberg, Homological Algebra, Princeton, 1956

[C1] P. Cartier, Groups formels associés aux anneaux de Witt généralisés, C.R. Acad. Sci. Paris 265(1967), 49–52.

[C2] P. Cartier, Modules associés à un groupe formel commutatif. Courbes typiques, C.R. Acad. Sci. Paris 265(1967), 129–132.

[D] B. Dayton, The Picard group of a reduced G-algebra, JPAA 59(1989), 237–253.

[DW] B. Dayton and C. Weibel, On the naturality of Pic, SK_0 and SK_1, in J.Jardine and V. Snaith (eds.), Algebraic K-theory: Connections with Geometry and Topology, pp.1–28, Kluwer Acad. Publishers, 1989.

[DI] R.K. Dennis and K. Igusa, Hochshild Homology and the second obstruction for Pseudoisotopy, Lecture Notes in Math 966, pp.7–58, Springer -Verlag, 1982.

[FT] B. Feigin and B. Tsygan, Additive K-theory, Lecture Notes in Math 1289, pp. 67–209, Springer Verlag, 1987.

[GRW] S. Geller, L. Reid, and C.Weibel, The cyclic homology and K-theory of curves, J. reine angew. Math. 393(1989), 39–90.

[Goodw] T. Goodwillie, Cyclic homology, derivations and the free loop space., Top. 24(1985), 187–215.

[Grays] D. Grayson, Grothendieck rings and Witt vectors, Comm. Alg. 6(1978), 249–255.

[Haz] M. Hazewinkel, Three Lectures on Formal Groups, in Lie Algebras and related topics, CMS Conference Proceedings Vol. 5, pp. 51–67, AMS, 1986.

[HKR] G. Hochschild, B. Konstant and A. Rosenberg, Differential forms on regular affine algebras, Trans. AMS 102(1962), 383–408.

[HK] D. Husemoller and C. Kassel, Cyclic Homology, to appear.

[Igusa] K. Igusa, What happens to Hatcher and Wagoner's etc., Lecture Notes in Math 1046, pp. 105–172, Springer Verlag, 1984.

[K-CCM] C. Kassel, Cyclic Homology, Comodules, and Mixed complexes, J. Alg 107(1987), 195–216.

[K-CCB] C. Kassel, Caractère de Chern bivariant, K-theory 3(1989), 367–400.

[K-LP] C. Kassel, Homologie cyclique, charactère de Chern et lemme de perturbation, J. reine angew Math. 408(1990), 159–180.

[Kunz] E. Kunz, Kähler Differentials, Vieweg, 1986

[LR] J. Labute and P. Russell, On K_2 of truncated polynomial rings, JPAA 6(1975), p.239–251.

[Lang] S. Lang, Algebra, Addison-Wesley, 1965.

[Laz] M. Lazard, Commutative Formal Groups, Lect. Notes in Math. 443, Springer-Verlag, 1975.

[LQ] J.-L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, Comm. Math. Helv. 59(1984), 565–591.

[Mac] S. MacLane, Homology, Springer-Verlag, 1963.

[Mats] H. Matsumura, Commutative Algebra, Benjamin, 1970

[Rine] G. Rinehart, Differential forms on general commutative algebra, Trans. AMS 108(1963), 195–222.

[Rob] L. Roberts, Kähler differentials and HC_1 of certain graded K–algebras, in J. Jardine and V. Snaith (eds.), Algebraic K–theory: Connections with Geometry and Topology, pp.389–424, Kluwer Acad. Publ. 1989.

[S-TP] J. Stienstra, On K_2 and K_3 of truncated polynomial rings, in Lect. Notes in Math. 854, pp. 409–455, Springer-Verlag, 1981.

[S-OE] J. Stienstra, Operations in the higher K-theory of endomorphisms, CMS Conf. Proc., Vol 2 (part 2) (1982), 59–115.

[SvdK] J. Stienstra and W. van der Kallen, The relative K_2 of truncated polynomial rings, JPAA 34(1984) 277–289.

[W-MVS] C.A. Weibel, Mayer-Vietoris sequences and module structures on NK_*, in Lect. Notes. in Math. 966, pp. 466–493, Springer-Verlag, 1982.

[W-MGR] C.A. Weibel, Module Stuctures on the K-theory of Graded Rings, J. Alg. 105(1987), 465–483.

[W-LL] C. Weibel, Appendix to Kähler differentials and HC_1 of certain graded K-algebras by Leslie G. Roberts in J. Jardine and V. Snaith (eds.), Algebraic K-theory: Connections with Geometry and Topology, pp. 389–424, Kluwer Acad. Publ. 1989.

[WG] C. Weibel and S. Geller, Étale descent for Hochschild and Cyclic homology, Comm. Math. Helv. 66 (1991), 368-388.