Chapter 6

Analysis

Elliptic Functions

In this chapter we discuss Elliptic functions, a very important part of classical mathematics that is usually overlooked at the undergraduate level. We concentrate on the Jacobi Elliptic functions which, while quite important in pure and applied mathematics in the nineteenth century have been almost forgotten today. While not part of the classical Theory of Equations course their inclusion is consistent with one theme of this book – introducing classical mathematics in a modern context. In addition, we will apply some of the theory of cubic and biquadratic functions that we have learned earlier in the course.

6.1 Trigonometric and Hyperbolic Functions

The Jacobi elliptic functions were developed to solve problems that could not be solved by the trigonmetic and hyperbolic functions. The reader will have a better appreciation of the rest of this chapter if she or he reviews these functions. I will remind you of some of the definitions but mostly I will give an exercise set to help you review.

We will be working with the trigonometric functions $\sin, \cos, \tan$ and sec. Your author is in sympathy with those who advocate not introducing young mathematics students to the cotangent and cosecant but feels that the secant is important enough a function to have its own name. But we will be equally interested in the inverse functions to these functions, especially the inverse sine $\sin^{-1}$ and the inverse tangent $\tan^{-1}$. We use the standard but inconsistent convention that $\cos^2(x)$ means $(\cos(x))^2$ but $\cos^{-1}(x)$ is the inverse function to cos, not the reciprocal $\frac{1}{\cos x} = \sec x$. Of special interest are the differentiation formulas for the inverse functions, in particular the derivatives are rational or algebraic functions, not trigonometric functions. You should also review the many relations between the various trigonometric functions or inverse functions. In particular any of the inverse functions can be expressed as an algebraic expression in any of the others.

Note that the trigonmetric functions can be extended to functions defined on the entire complex plane. The formulas are

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

where the complex exponential fuction can be defined by the usual power series or in terms of the real trigonometric functions by

$$e^{x+iy} = e^x(\cos y + i \sin y)$$

The definition of the complex trigonometric functions can motivate the hyperbolic functions, mainly

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

with $\tanh x = \sinh(x)/\cosh(x)$ and $\text{sech}(x) = 1/\cosh(x)$. Again you should recall the relations between these such as $\cosh^2 x - \sinh^2 x = 1$, from which the name "hyperbolic" arises, and the differentiation formulas. The hyperbolic functions have inverse functions also. While you may work with them directly, it turns out that the inverse hyperbolic functions can be expresed in terms of the logarithm function.

Exercise 1 With this introduction, the reader is invited to try the following exercise set as a warmup for this chapter.

1. Show $\sin^2 z + cos^2 z = 1$ for all complex z
2. Describe the function $f(x) = i \sin(ix)$ for x real.
3. Show $\sin^{-1}(x) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

4. Find $\tan(\cos^{-1}(x))$

5. Find $\frac{d}{dx}(\sqrt{x})$

6. Verify the formula for $x > a > b$

$$\int \frac{dx}{(x-b)\sqrt{x-a}} = \frac{2}{\sqrt{a-b}} \sin^{-1}\left(\sqrt{\frac{x-a}{x-b}}\right) + C$$

Hint: Differentiate both sides.

7. Verify the formula for $a > b > x > c$

$$\int \frac{dx}{(a-x)\sqrt{(b-x)(x-c)}} = \frac{2}{\sqrt{(a-b)(a-c)}} \sin^{-1}\left(\sqrt{\frac{(a-b)(x-c)}{(b-c)(a-x)}}\right)+C$$

8. Show $1 - \tanh^2 x = \text{sech}^2 x$ for all real x

9. Describe the function $g(x) = \cosh(ix)$ for real x.

10. Show $\sinh^{-1}(x) = \cosh^{-1}(\sqrt{1+x^2})$

11. Show $\sinh(\tanh^{-1}(x)) = \frac{x}{\sqrt{1-x^2}}$

12. Given $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ show $\text{sech}^{-1}(x) = \ln(1+\sqrt{1-x^2}) - \ln x$

13. When $a > x > b$ verify the formula

$$\int \frac{dx}{(x-b)\sqrt{a-x}} = \frac{-2}{\sqrt{a-b}} \sinh^{-1}\left(\sqrt{\frac{a-x}{x-b}}\right) + C$$

14. For $a > b > c > x$ verify

$$\int \frac{dx}{(a-x)\sqrt{(b-x)(c-x)}} = \frac{2}{\sqrt{(a-b)(a-c)}} \sinh^{-1}\left(\sqrt{\frac{(a-b)(c-x)}{(b-c)(a-x)}}\right)+C$$

6.2 The Historical Background

It would be nice if we could say that the Jacobi elliptic functions parameterize the ellipse the same way the trigonometric functions parameterize the circle, i.e. a constant speed parameterization. But this is not true. The term "elliptic" is actually not very descriptive but is used in deference to historical precedents. One possible explaination is that the elliptic functions form a set of functions with properties like the trigonometric and hyperbolic functions, but more like the trigonmetric. The name "elliptic" may be used simply as meaning "not hyperbolic". In this section I will try to give another explaination of how this term perhaps came to be used.

The 18^{th} century saw the development of the calculus, primarily by the followers of Leibnitz such as the Bernoullis and Euler. They developed all the "standard" integration techniques using power, rational, exponential, logarithmic and trigonometric functions. In particular they may have been amazed to find that integrals involving square roots of quadratic functions often were expressed in terms of the inverse trigonometric and hyperbolic functions. One problem they could not solve was the calculation of the arc length of an arc of an ellipse.

To understand this, start with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Assume, for the sake of argument, that $b > a > 0$, i.e. the major axis is along the y-axis and the minor axis is along the x-axis. This can be parameterized by $x = a\cos\phi$ and $y = b\sin\phi$. From calculus the arc length (say for $0 \leq \phi \leq u$) is given by

$$s = \int_0^u \sqrt{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2}\, d\phi = \int_0^u \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\, d\phi$$

Now we let $\kappa = \sqrt{\frac{b^2-a^2}{b^2}}$, this κ is called the eccentricity of the ellipse (in analytic geometry usually denoted by e). Then the Pythagorean identity gives $b^2\cos^2\phi = b^2 - b^2\sin^2$ so $a^2\sin^2\phi + b^2\cos^2\phi = a^2\sin^2\phi + (b^2 - b^2\sin^2\phi) = b^2 + (a^2-b^2)\sin\phi = a^2(1 - \frac{b^2-a^2}{b^2}\sin^2\phi) = b^2(1-\kappa^2\sin^2\phi)$. Hence the integral for arc length becomes

$$s = b\int_0^u \sqrt{1-\kappa^2\sin^2\phi}\, d\phi$$

But this was not a function that they knew how to integrate.

Another class of integrals that seemed intractible using the standard functions of calculus were integrals of the form

$$\int \frac{S + T\sqrt{Y}}{U + V\sqrt{Y}} dx$$

where S, T, U, V are rational functions of x and Y is a polynomial in x of degree three or four. A. Legendre gave the first systematic study of these integrals and showed that they could be expressed as sums of standard integrals and integrals of the form

$$F = \int \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}}$$
$$E = \int \sqrt{\frac{1-\kappa^2 x^2}{1-x^2}}\, dx$$
$$\Pi = \int \frac{dx}{(1-\eta x^2)\sqrt{(1-x^2)(1-\kappa^2 x^2)}}$$

for appropriate constants η, κ.

A simple trigonometric substitution gives

$$F = \int \frac{du}{\sqrt{1-\kappa^2 \sin^2 u}}$$
$$E = \int \sqrt{1-\kappa^2 \sin^2 u}\, du$$
$$\Pi = \int \frac{du}{(1+\eta \sin^2 u)\sqrt{1-\kappa^2 \sin^2 u}}$$

We note that the trigonometric form of E is just the indefinite form of the integral for the arc length of the ellipse. Perhaps this is the reason that the name "elliptic" came to be used. In any case the integrals F, E, Π became known as "Legendre's canonical incomplete elliptic integrals of the first, second and third kinds" respectively.

It turns out that for the development of elliptic functions that the Legendre integral of the first kind, which appears to have little to do with the ellipse, played the major role. It was Abel who suggested that in the integral of the first kind

$$t = F(\phi) = \int_0^\phi \frac{du}{\sqrt{1-\kappa^2 \sin^2 u}} \tag{6.1}$$

one should look at not the function $t = F(\phi)$ but rather the inverse function which would give ϕ as a function of t. This, he asserted would be the more interesting function to study. For example, he noted, that if $\kappa = 0$ in the non-trigonometric form of this integral one has

$$t = \int_0^\phi \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}\phi$$

which has inverse function $\phi = \sin t$.

From this idea mathematicians such as Liouville, Jacobi, Weierstrass and Riemann were able, over the course of the 19^{th} century, to build an elaborate theory of elliptic functions and to connect this theory back to the geometric properties of the curve Y, or more precisely, the curves $y^2 = f(x)$ where $f(x)$ is a polynomial of degree 3 or 4.

By the Fundamental Theorem of Calculus differentiating both sides of (6.1) with respect to ϕ gives

$$\frac{dt}{d\phi} = \frac{1}{\sqrt{1-\kappa^2\sin^2\phi}}$$

We want the inverse function. But the derivative of the inverse function is just the reciprocal of the derivative (but evaluated at a different place, of course). So from this and the fact that $t = 0$ when $\phi = 0$ we arrive at the differential equation for the inverse function:

$$\frac{d\phi}{dt} = \sqrt{1-\kappa^2\sin^2\phi}, \quad \phi(0) = 0$$

This will be our starting point for the development of the Jacobi elliptic functions.

6.3 The Jacobi Elliptic Functions

Based on the discussion in the preceeding section, we set κ so that $0 \le \kappa < 1$. The functions we define in this section all depend on the choice of κ but, to avoid unecessarily confusing notation, we will not specifically use κ as part of the notation. Using the previous section as motivation, we now make the key definition.

Definition 6.3.1 *The function $\phi = \phi(t)$ is the solution of the following differential equation:*

$$\frac{d\phi}{dt} = \sqrt{1-\kappa^2\sin^2\phi}, \quad \phi(0) = 0 \tag{6.2}$$

Figure 6.1: The function ϕ or am

Using theorems of Differential Equations one can show that this function exists, is defined for all real t and, by an easy argument, that this definition gives a unique function $\phi(t)$. This function is known as the "amplitude" function and is sometimes denoted by $\mathrm{am}(t)$. While this is for us a new function, not expressible in terms of the functions we already know, it is easily computed or graphed from the defining differential equation. Basically, for κ small the function ϕ is essentially the identity function, i.e. the linear function with slope 1 and intercept 0. As κ gets close to 1 then the function ϕ grows more slowly in sort of a wavy fashion. Figure 6.1 gives some examples.

You can use Maple or even the Differential Equation Solver on the TI-85 to graph this function. Because it is such a smooth function the differential equation is very easy to solve numerically. Here is one simple method that works well if you want to find actual values of the function – the Runge-Kutta Midpoint Method. It can be implemented in MAPLE, MATLAB, the TI-85 etc. I will show below that ϕ is "quasi periodic", i.e. locally repeats itself up to a constant, and as the functions that will be defined in terms of ϕ will be periodic it is enough to know $\phi(t)$ for relatively small positive values of t.

Algorithm 1 To calculate $\phi(t)$ for some $t > 0$. Choose integer n so that $n \geq 10t$ and let $h = t/n \leq 0.1$ which will be a sufficiently small increment. Define the function $f(x) = \sqrt{1 - \kappa^2 \sin^2 x}$ and create a sequence $y_0, y_1, \ldots, y_n$ recursively by

$$y_0 = 0$$
$$y_{j+1} = y_j + hf(y_j + \frac{h}{2} f(y_j))$$

Then $\phi(t) = y_n$ Note you also have calculated $\phi(jh) = y_j$ for $j = 1, \ldots, n$. Note also that the formula is not a misprint, there are nested evaluations of f.

We can now define the main objects of this chapter:

Definition 6.3.2 *Assume κ has been given, $0 \leq \kappa < 1$. Then the* **Jacobi elliptic functions** sn*,* cn *and* dn *are defined by*

$$\begin{aligned} \mathrm{sn}(t) &= \sin \phi(t) \\ \mathrm{cn}(t) &= \cos \phi(t) \\ \mathrm{dn}(t) &= \sqrt{1-\kappa^2 \sin^2 \phi(t)} = \sqrt{1-\kappa^2 sn^2(t)} \end{aligned}$$

It is traditional to pronounce these by pronouncing the separate letters, for example $\mathrm{sn}(t)$ is pronounced "s n t"

When $\kappa = 0$ then $\phi(t) = t$ so sn, cn are just the usual trigonometric sine and cosine and dn is the constant 1 function. For small κ sn and cn are only slight deformations, basically a stretching, of the sine and cosine functions. By small κ we mean $\kappa < 0.6$ or so. As κ increases the deformations become more pronounced and dn approaches cn. In the limit as κ goes to 1, cn and dn approach sech, the hyperbolic secant, while sn approaches the hyperbolic tangent.

The author has a Java applet on his web site that you can use to see how the shape of these functions varies with κ.

Like the trignonometric functions, the elliptic functions satisfy several identities. For example the basic ones are

$$\begin{aligned} \mathrm{cn}^2 + \mathrm{sn}^2 &= 1 \\ \mathrm{dn}^2 + \kappa^2 \mathrm{sn}^2 &= 1 \\ \mathrm{dn}^2 - \kappa^2 \mathrm{cn}^2 &= \breve{\kappa}^2 \end{aligned}$$

where $\breve{\kappa}$ is the complementary modulus defined by

$$\kappa^2 + \breve{\kappa}^2 = 1$$

Although, as explained in the previous section, the Jacobi Elliptic functions have little to do with the ellipse it is instructive to note that they can be used to parameterize the ellipse, similar to the trigonometric functions. Again consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $0 < b < a$ and eccentricity $\kappa = \sqrt{\frac{a^2-b^2}{a^2}}$. Then $x = a\mathrm{cn}(t), y = b\mathrm{sn}(t)$ is a parameterization of the ellipse by the first identity above. Note carefully, however,

Figure 6.2: The Jacobi elliptic functions $\kappa = .866$

that t is **not** the angle nor is this a unit speed parameterization. However it is instructive to note that the distance from the point $(a\mathrm{cn}(t), b\mathrm{sn}(t))$ is

$$\sqrt{(a\mathrm{cn}(t))^2 + (b\mathrm{sn}(t))^2} = \sqrt{a^2 - a^2\mathrm{sn}^2(t) + b^2\mathrm{sn}^2(t)} =$$
$$\sqrt{a^2(1 - \left(\frac{a^2 - b^2}{a^2}\right)\mathrm{sn}^2(t)} = a\sqrt{1 - \kappa^2\mathrm{sn}^2(t)} = a\mathrm{dn}(t)$$

Thus $a$dn plays the role of the constant radius r for the circular functions. In fact $a$dn varies between a and b for this κ.

Like the trigonometric functions we can form new functions by forming reciprocals or quotients of the original functions. Traditionally $\frac{1}{\mathrm{cn}}$ is denoted nc, 1/sn is ns and 1/dn by nd. It is allowable to write $\frac{\mathrm{sn}}{\mathrm{cn}}$ as sc but, in analogy with the tangent, it is usually written tn and cn/sn is written ctn. The mathematician Clifford went so far as to suggest a similar method of denoting the hyperbolic and circular functions, writing sinh, cosh and tanh as sh, ch and th respectively and just c, s and t for the cosine, sine and tangent. The hyperbolic secant could be hc but what would we use for the secant? Needless to say, this notation did not catch on.

As mentioned above, ϕ is quasi-periodic. This follows directly from the defining differential equation $\frac{d\phi}{dt} = \sqrt{1 - \kappa^2 \sin^2 \phi}$. First of all since the square root is always positive (we assume $\kappa < 1$) ϕ is an increasing function with derivative bounded below by $\breve{\kappa} = \sqrt{1 - \kappa^2} > 0$. Thus as $t \to \infty$ we have $\phi(t) \to \infty$. By the intermediate value theorem we have some number $2K > 0$ so that $\phi(2K) = \pi$.

But $\sin^2 x$ is periodic of period π so the differential equation

$$\frac{d\phi}{dt} = \sqrt{1 - \kappa^2 \sin^2(\phi + \pi)}, \quad \phi(0) = \pi$$

is identical to the original equation except for initial value. Thus the differece between the two solutions is is the constant π. Of course, it then follows that $\phi(4K) = 2\pi$, and so on. So specifically, what quasi-periodic means here is that if $t = 2nK + x$ where $0 \le x < 2K$ then $\phi(t) = \phi(x) + n\pi$.

It follows immediately from the last paragraph that cn and sn are periodic of period $4K$ since sine and cosine are periodic of period 2π while dn is periodic of period $2K$ since it is a function of $\sin^2$ which is periodic of period π.

The question remains, what is K? By taking the reciprocal of both sides of the defining differential equation (or returning to the motivation in the previous section) we have

$$\frac{dt}{d\phi} = \frac{1}{\sqrt{1 - \kappa^2 \sin^2 \phi}}$$

where now t is the inverse function of ϕ. But this equation is easily integrated from

$$dt = \frac{1}{\sqrt{1 - \kappa^2 \sin^2 \phi}} d\phi$$

giving

$$t = \int_0^{\phi} \frac{1}{\sqrt{1 - \kappa^2 \sin^2 x}}\, dx$$

Substituting π for ϕ would give $t = 2K$ but exploiting the symmetry of the integrand it is evident that this is twice the integral

$$K = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \kappa^2 \sin^2 x}}\, dx \tag{6.3}$$

The latter integral is known as "Legendre's complete elliptic integral of the first kind" and is implemented under the name

$\mathsf{LegendreKc}(\kappa)$

in MAPLE. Actually, MAPLE's $\mathsf{LegendreF}(x, \kappa)$ is the inverse function of sn so also $K = \mathsf{LegendreF}(1, \kappa)$.

As an example, when $\kappa = .866 \approx \frac{\sqrt{3}}{2}$ then $K \approx 2.1565$ so, as shown in Figure 2, the period of cn and sn is $4k \approx 8.6261$, while dn has period 4.3130. Note that as κ approaches 1 the period approaches infinity and, in fact, the limiting functions sech and tanh are not periodic.

We end this section by calculating the derivatives of the Jacobi Elliptic functions. The defining differential equation tells us that the derivative $\frac{d\phi}{dt} = \mathrm{dn}(t)$ (compare definitions) so a simple application of the chain rule gives

$$\frac{d\mathrm{cn}(t)}{dt} = \frac{d\cos(\phi(t))}{dt} = -\sin(\phi(t))\frac{d\phi(t)}{dt} = -\mathrm{sn}(t)\mathrm{dn}(t)$$
$$\frac{d\mathrm{sn}(t)}{dt} = \frac{d\sin(\phi(t))}{dt} = \cos(\phi(t))\frac{d\phi(t)}{dt} = \mathrm{cn}(t)\mathrm{dn}(t)$$

Again we see that when $\kappa = 0$ the special case is just the usual trigonometric cosine and sine since dn is then the constant 1. For dn we get

$$\frac{d\mathrm{dn}(t)}{dt} = \frac{d}{dt}\sqrt{1-\kappa^2\sin^2\phi(t)} = \frac{1}{2\sqrt{1-\kappa^2\sin^2\phi(t)}}\frac{d(1-\kappa^2\sin^2\phi(t))}{dt} =$$
$$\frac{1}{2\sqrt{1-\kappa^2\sin^2\phi(t)}}(-2\kappa^2\sin\phi(t)\cos\phi(t))\frac{d\phi(t)}{dt} = -\frac{\kappa^2\mathrm{sn}(t)\mathrm{cn}(t)}{\mathrm{dn}(t)}\mathrm{dn}(t) =$$
$$-\kappa^2\mathrm{sn}(t)\mathrm{cn}(t)$$

And for the elliptic tangent $\mathrm{tn} = \frac{\mathrm{sn}}{\mathrm{cn}}$ we have

$$\frac{d\mathrm{tn(t)}}{dt} = \frac{d}{dt}\frac{\mathrm{sn(t)}}{\mathrm{cn(t)}} = \frac{\mathrm{cn}(t)\mathrm{dn}(t)\mathrm{cn}(t) - \mathrm{sn}(t)(-\mathrm{sn}(t)\mathrm{dn}(t)}{\mathrm{cn}^2(t)} =$$
$$\frac{(\mathrm{cn}^2(t)+\mathrm{sn}^2(t))\mathrm{dn}(t)}{\mathrm{cn}^2(t)} = \frac{\mathrm{dn}(t)}{\mathrm{cn}^2(t)}$$

6.4 The inverse Jacobi Elliptic Functions

Since the Jacobi elliptic functions arose as inverses of some integrals, it is the original integrals, that is the inverses of the Jacobi functions, that we are most interested in.

As usual, assume the modulus κ has been set, $\breve{\kappa}$ is the complementary modulus and K is given by (6.3). The domain and range for the inverse elliptic functions is motivated by Figure 6.2 and the discussion on the quasi-periodicity of ϕ. The domain for the inverse functions $\mathrm{sn}^{-1}(x)$ and $\mathrm{cn}^{-1}(x)$ is $-1 \leq x \leq 1$ which is the range of sn and cn. Since sn and dn are not 1–1 functions we have to arbitrarily choose a range, we will choose $-K \leq \mathrm{sn}^{-1}(x) \leq K$ and $0 \leq \mathrm{cn}(x) \leq 2K$. For dn^{-1} we have domain $\breve{\kappa} \leq x \leq 1$ and will arbitrarily choose range $0 \leq \mathrm{dn}^{-1}(x) \leq K$. Finally tn^{-1} will have the entire real line as domain but range $-K < \mathrm{tn}^1 < K$.

What we want to find are the differentiation formulas for these functions. Our tools will be the basic identities

$$\mathrm{cn}^2(t) + \mathrm{sn}^2(t) = 1 \tag{6.4}$$

$$\mathrm{dn}^2(t) + \kappa^2\mathrm{sn}^2(t) = 1 \tag{6.5}$$

$$\mathrm{dn}^2(t) - \kappa^2\mathrm{cn}^2(t) = \breve{\kappa}^2 \tag{6.6}$$

and the differentiation formulas

$$\frac{d\mathrm{cn}(t)}{dt} = -\mathrm{sn}(t)\mathrm{dn}(t) \tag{6.7}$$

$$\frac{d\mathrm{sn}(t)}{dt} = \mathrm{cn}(t)\mathrm{dn}(t) \tag{6.8}$$

$$\frac{d\mathrm{dn}(t)}{dt} = -\kappa^2\mathrm{sn}(t)\mathrm{cn}(t) \tag{6.9}$$

$$\frac{d\mathrm{tn(t)}}{dt} = \frac{\mathrm{dn}(t)}{\mathrm{cn}^2(t)} \tag{6.10}$$

I will calculate $\frac{d\mathrm{cn}^{-1}(x)}{dx}$ to illustrate the method. I first set $x = \mathrm{cn}(t)$ so that $t = \mathrm{cn}^{-1}(x)$ The trick is to calculate $\mathrm{sn}(t)$ and $\mathrm{dn}(t)$ in terms of x. To this end formula (6.4) gives $\mathrm{sn}^2(t) + x^2 = 1$ so $\mathrm{sn}^2(t) = 1 - x^2$ or $\mathrm{sn}(t) = \sqrt{1 - x^2}$ Likewise (6.6) gives $\mathrm{dn}^2(t) - \kappa^2 x^2 = \breve{\kappa}^2$ so $\mathrm{dn}(t) = \sqrt{\breve{\kappa}^2 + \kappa^2 x^2}$. Now from formula (6.7) above $\frac{d\mathrm{cn}(t)}{dt} = -\mathrm{sn}(t)\mathrm{dn}(t) = -\sqrt{1 - x^2}\sqrt{\breve{\kappa}^2 + \kappa^2 x^2}$. Taking the reciprocal of both sides we get

$$\frac{d\mathrm{cn}^{-1}(x)}{dx} = \frac{dt}{dx} = \frac{-1}{\sqrt{(\breve{\kappa}^2 + \kappa^2 x^2)(1 - x^2)}}$$

.

Exercise 2 Using the technique above verify the following formulas:

a) $$\frac{d\mathrm{sn}^{-1}(x)}{dx} = \frac{1}{\sqrt{(1 - \kappa^2 x^2)(1 - x^2)}}$$

b) $\dfrac{d\mathrm{dn}^{-1}(x)}{dx} = \dfrac{-1}{\sqrt{(1-x^2)(x^2-\breve{\kappa}^2)}}$

c) $\dfrac{d\mathrm{tn}^{-1}(x)}{dx} = \dfrac{1}{\sqrt{(1+x^2)(1+\breve{\kappa}^2 x^2)}}$

Hint for c): This is tricky as you have to figure out how to find $\mathrm{dn}(t)$ and $\mathrm{cn}(t)$ in terms of $x = \mathrm{tn}(t)$.

With the inverse trigonometric functions, if you know one you can calculate the others. This is true also of the inverse elliptic functions. There are several reasons why this is useful. One is that you may have access to only one. For example MAPLE implements the function

$$\mathrm{sn}^{-1}(x) = \mathsf{LegendreF}(x, \kappa)$$

for $0 \le x \le 1$ and modulus κ. It does not implement the other inverse elliptic functions. Thus one way to find, say, $\mathrm{dn}^{-1}(x)$ would be to use the formula (derived below) $\mathrm{dn}^{-1}(x) = \mathrm{sn}^{-1}\left(\sqrt{\frac{1-x^2}{\kappa^2}}\right)$. Of course, another way to solve this problem is to integrate numerically the derivative of dn^{-1} above – see the next section.

A more important reason for conversion is that sometimes formulas obtained using one inverse function have a much nicer form using another. Finally, one may need to do a calculation such as

$$\mathrm{sn}(\mathrm{dn}^{-1}(x)) = \mathrm{sn}\Big(\mathrm{sn}^{-1}\left(\sqrt{\frac{1-x^2}{\kappa^2}}\right)\Big) = \sqrt{\frac{1-x^2}{\kappa^2}}$$

The formula $\mathrm{dn}^{-1}(x) = \mathrm{sn}^{-1}\left(\sqrt{\frac{1-x^2}{\kappa^2}}\right)$ mentioned above may be derived as follows: Write $\mathrm{dn}^{-1}(x) = t$ so that $\mathrm{dn}(t) = x$. From formula (6.5) we have $x^2 + \kappa^2 \mathrm{sn}^2(t) = 1$ so $\mathrm{sn}(t) = \sqrt{\frac{1-x^2}{\kappa^2}}$ Then applying sn^{-1} to both sides we have

$$\mathrm{dn}^{-1}(x) = t = \mathrm{sn}^{-1}\left(\sqrt{\frac{1-x^2}{\kappa^2}}\right)$$

Exercise 3 Verify the following conversions:

a) $\mathrm{sn}^{-1}(x) = \mathrm{cn}^{-1}(\sqrt{1-x^2}) = \mathrm{dn}^{-1}(\sqrt{1-\kappa^2 x^2})$

b) $\mathrm{cn}^{-1}(x) = \mathrm{sn}^{-1}(\sqrt{1-x^2}) = \mathrm{dn}^{-1}(\sqrt{\breve{\kappa}^2 + \kappa^2 x^2})$

c) $\mathrm{dn}^{-1}(x) = \mathrm{sn}^{-1}\left(\sqrt{\frac{1-x^2}{\kappa^2}}\right) = \mathrm{cn}^{-1}\left(\sqrt{\frac{\breve{\kappa}^2 - x^2}{\kappa^2}}\right)$

The following are a bit harder:

d) Express the inverse functions above in terms of tn^{-1}

e) Express $\mathrm{tn}^{-1}(x)$ in terms of the other 3 inverse elliptic functions.

6.5 Elliptic Integrals

In this section we will use more or less traditional techniques of integration to show that all integrals of the form $\int f(x)^{-\frac{1}{2}}\, dx$ where $f(x)$ is a cubic or biquadratic polynomial can be expressed in terms of the Jacobi elliptic functions.

The formulas in the previous section for the derivatives of the inverse Jacobi elliptic functions immediately give us the following indefinite integration formulas:

$$\int \frac{dx}{\sqrt{(1-\kappa^2 x^2)(1-x^2)}} = \mathrm{sn}^{-1}(x) + C \tag{6.11}$$

$$\int \frac{-dx}{\sqrt{(\breve{\kappa}^2 + \kappa^2 x^2)(1-x^2)}} = \mathrm{cn}^{-1}(x) + C \tag{6.12}$$

$$\int \frac{-dx}{\sqrt{(1-x^2)(x^2-\breve{\kappa}^2)}} = \mathrm{dn}^{-1}(x) + C \tag{6.13}$$

$$\int \frac{dx}{\sqrt{(1+x^2)(1+\breve{\kappa}^2 x^2)}} = \mathrm{tn}^{-1}(x) + C \tag{6.14}$$

These formulas are valid where both sides make sense.

The quartics here are all even (symmetric about the y-axis). One can use a change of variables and appropriate choice of κ to derive other integrals of even quartics. Alternatively, given the formula it can be verified by differentiating both sides.

Example 6.5.1 To verify the formula (for $a > b > x$)

$$\int \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a}\mathrm{sn}^{-1}(\frac{x}{b}) + C, \quad \kappa = \frac{b}{a} \tag{6.15}$$

one differentiates the right hand side

$$\frac{d}{dx}\frac{1}{a}\mathrm{sn}^{-1}(\frac{x}{b}) = \frac{1}{\sqrt{(1-(\frac{b}{a})^2(\frac{x}{b})^2)(1-(\frac{x}{b})^2)}}\frac{1}{b} =$$

$$\frac{1}{\sqrt{a^2(1-\frac{x^2}{a^2})(1-\frac{x^2}{b^2})b^2}} = \frac{1}{\sqrt{(a^2-x^2)(b^2-x^2)}}$$

Exercise 4 Verify the following integration formulas by differentiating the right hand side. If you are more ambitious you might try doing these by substitution. Indicate the restrictions on a, b and x.

$$\int \frac{dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{-1}{\sqrt{a^2+b^2}}\mathrm{cn}^{-1}\left(\frac{x}{b}\right) + C, \qquad \kappa = \frac{b}{\sqrt{a^2+b^2}} \tag{6.16}$$

$$\int \frac{dx}{\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\mathrm{cn}^{-1}\left(\frac{b}{x}\right) + C, \qquad \kappa = \frac{a}{\sqrt{a^2+b^2}} \tag{6.17}$$

$$\int \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{-1}{a}\mathrm{dn}^{-1}\left(\frac{x}{a}\right) + C, \qquad \kappa = \sqrt{1-\frac{b^2}{a^2}} \tag{6.18}$$

$$\int \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a}\mathrm{tn}^{-1}\left(\frac{x}{b}\right) + C, \qquad \kappa = \sqrt{1-\frac{b^2}{a^2}} \tag{6.19}$$

Exercise 5 Using the Jacobi elliptic functions, find the indefinite integral

$$\int \frac{dx}{\sqrt{x^4-1}} \qquad \text{for } x > 1$$

So far the integrands have involved biquadratics. But notice what happens with the following differentiation:

$$\frac{d}{dx}2\mathrm{sn}^{-1}(\sqrt{x}) = \frac{2}{\sqrt{(1-\kappa^2\sqrt{x}^2)(1-\sqrt{x}^2)}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{\sqrt{x(1-\kappa^2 x)(1-x)}}$$

A similar differentiation (see the next exercise) gives

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}}\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\gamma}{x-\gamma}}\right) \qquad \kappa = \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for $x > \alpha > \beta > \gamma$. Here we might wish for a different domain for x. If $\alpha > x > \beta$ or $\gamma > x$ then $(x-\alpha)(x-\beta)(x-\gamma)$ would be negative so the integral would not make sense (as a real integral) but we could do

$$\int \frac{dx}{\sqrt{-(x-\alpha)(x-\beta)(x-\gamma)}}$$

instead. When $\beta > x > \gamma$ the original integral makes sense but $\sqrt{\frac{\alpha-\gamma}{x-\gamma}} > 1$ so is not in the domain of sn^{-1}. Thus, we find, to completely give the integration formulas when $f(x)$ is a cubic with three real roots we need to give not just one formula but at least four. These formulas will be checked in the following exercise.

Exercise 6 Assume $\alpha > \beta > \gamma$ are three distinct real numbers. Let $\kappa_0 = \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}}$ Show that

$$\breve{\kappa}_0 = \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}$$

Now verify the following formulas:

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}}\mathrm{sn}^{-1}\left(\sqrt{\frac{x-\alpha}{x-\beta}}\right) + C, \quad \kappa = \kappa_0 \tag{6.20}$$

when $x > \alpha$

$$\int \frac{dx}{\sqrt{-(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{-2}{\sqrt{\alpha-\gamma}}\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-x}{\alpha-\beta}}\right) + C, \quad \kappa = \breve{\kappa}_0 \tag{6.21}$$

when $\alpha > x > \beta$

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}}\mathrm{sn}^{-1}\left(\sqrt{\frac{x-\gamma}{\beta-\gamma}}\right) + C, \quad \kappa = \kappa_0 \tag{6.22}$$

when $\beta > x > \gamma$

$$\int \frac{dx}{\sqrt{-(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}}\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\gamma}{\alpha-x}}\right) + C \quad \kappa = \breve{\kappa}_0 \tag{6.23}$$

when $\gamma > x$

It should be noted that because the formula depends on the location of x that MAPLE will not give indefinite forms of elliptic integrals. It will however give exact formulas for definite elliptic integrals in terms of the LegendreF function (which, as we have noted is sn^{-1}). But you are aware that there are many equivalent formulas for a definite integral so you won't be surprised when you do not recognize MAPLE's results. MAPLE uses a different approach to these integrals.

We also remark that if two (or more)roots are equal then this integral reduces to a standard integral. For example if $\alpha = \beta > \gamma$ then for $x > \alpha$ we have

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} = \int \frac{dx}{(x-\alpha)\sqrt{x-\gamma}} = \frac{-2}{\sqrt{\alpha-\gamma}}\sinh^{-1}\left(\sqrt{\frac{\alpha-\gamma}{x-\alpha}}\right)$$

These cases are known as "degenerate" cases. There are many such cases.

The exercise above shows how to find indefinite integrals of the form $\int f(x)^{-\frac{1}{2}}\,dx$ for all cubics $f(x)$ with distinct real roots. We will show how to derive the integrals for general cubics or biquadratics $f(x)$ from this, except in degenerate cases which can be handled by standard integration techniques. But first we need a result on rational functions.

Let $N = N(x) = ax^2+bx+c$ be a quadratic with no real roots, i.e. $b^2-4ac < 0$. Let $D = D(x)$ be any linear or quadratic function. Now let $y = y(x) = N(x)/D(x)$ be the quotient. We first note by algebra that $y' = \frac{N'D-ND'}{D^2}$ almost always has a quadratic numerator: in the case D is quadratic the cubic terms cancel, in the other case there are no cubic terms. In neither case would we expect the quadratic term to cancel. Now when D is quadratic the graph of $y = y(x)$ has a non-zero horizontal asymptote, no zeros and no vertical asymptotes or two of them. When D is linear then there is an oblique asymptote, no zeros and one vertical asymptote. Typical graphs appear in Figure 6.3. In each case it is seen that there must be at least one critical point, usually two. Thus the numerator of y' factors over the reals, usually as $\gamma(x-x_1)(x-x_2)$ where γ is a constant and x_1, x_2 are distinct real numbers. Now let $y_j = y(x_j), j = 0, 1$. Again, by inspection of the graphs the functions $y - y_j$ should have exactly one real zero at $x = x_j$. Thus we expect

$$y - y_1 = \frac{\alpha(x-x_1)^2}{D} \text{ and } y - y_2 = \frac{\beta(x-x_2)^2}{D}$$

Figure 6.3: Possibilities for $y = \frac{N}{D}$

for real constants α, β. It then follows that

$$y' = \frac{\gamma(x-x_1)(x-x_2)}{D^2} = \frac{\gamma}{\sqrt{|\alpha\beta|}D} \frac{\pm\sqrt{|\alpha|}(x-x_1)}{\sqrt{D}} \frac{\pm\sqrt{|\beta|}(x-x_2)}{\sqrt{D}} =$$

$$= \frac{\gamma}{\sqrt{|\alpha\beta|}D} \sqrt{\frac{\pm\alpha(x-x_1)^2}{D}} \sqrt{\frac{\pm\beta(x-x_2)^2}{D}} = M \frac{\sqrt{\pm(y-y_1)(y-y_2)}}{D}$$

where $M = \frac{\gamma}{\sqrt{|\alpha\beta|}}$ is a real constant. We have:

Lemma 6.5.2 *Let $y = N(x)/D(x)$ be a rational function where the numerator is a quadratic with no real roots and the denominator is linear or quadratic. Then, except in degenerate cases, there exist distinct real numbers y_1, y_2 and a real constant M so that*

$$y' = M \frac{\sqrt{\pm(y-y_1)(y-y_2)}}{D}$$

Actually this is more a definition than a lemma since by "degenerate cases" we mean those where the conclusion is false. Thus this Lemma really says that the conclusion is true except when it isn't and those cases we call degenerate. But the paragraph previous to the Lemma shows that we should expect the conclusion to be true most of the time.

We can now complete the calculation of integrals of the form $\int f(x)^{-\frac{1}{2}}\,dx$ Let $f(x)$ be a cubic or biquadratic polynomial with real coefficients. By D'Alembert's Theorem (Thm 1.9.9) $f(x)$ can be factored over the reals into a product of linear and irreducible quadratic polynomials. Recall that a quadratic is irreducible over the reals if it has no real roots.

We now treat the case that $f(x)$ has an irreducible quadratic factor $N(x)$. Then $f(x) = N(x)D(x)$ where D is linear or quadratic, possibly reducible. We use the change of variables $y = \frac{N}{D}$. By the Lemma above, except in degenerate cases, $\frac{dy}{dx} = C\frac{\sqrt{\pm(y-y_1)(y-y_2)}}{D}$ so

$$dx = \frac{1}{M}\frac{D}{\sqrt{\pm(y-y_1)(y-y_2)}}\,dy$$

Then

$$\int \frac{dx}{\sqrt{f(x)}} = \int \frac{dx}{\sqrt{D\frac{N}{D}}} = \int \frac{dx}{D\sqrt{y}} =$$

$$\int \frac{\frac{1}{M}\frac{D}{\sqrt{\pm(y-y_1)(y-y_2)}}/,dy}{D\sqrt{y}} = \frac{1}{M}\int \frac{dy}{\sqrt{\pm y(y-y_1)(y-y_2)}}$$

Thus we have reduced this calculation to the calculation of $\int g(x)^{-\frac{1}{2}}\,dx$ discussed earlier, where $g(x)$ has 3 real roots.

We have now shown how to calculate $\int f(x)^{-\frac{1}{2}}\,dx$ for **all** cubic $f(x)$ and some biquadratics. In fact, the only non-degerate case left is when $f(x)$ has four distinct real roots. It can be shown that these integrals all have the form $M\mathrm{sn}^{-1}\left(\sqrt{\frac{ax+b}{cx+d}}\right)$ for suitable constants a, b, c, d, M and κ depending on the interval of integration. However we would like to show that, as in the previous biquadratic with irreducible quadratic factor case , this case also reduces to the cubic case. So then, all the biquadratic non-degenerate cases reduce to the cubic case.

More generally, suppose $f(x)$ is a quartic with at least one real root α. Then, over the reals, $f(x) = (x-a)g(x)$ where $g(x)$ is a real cubic. We make the substitution $y = 1/(x-\alpha)$ or $(x-\alpha) = 1/y$ which gives $x = 1/y + \alpha$ So

$$dx = \frac{-1}{y^2}dy \text{ and } f(x) = f\left(\frac{1+\alpha y}{y}\right) = \frac{h(y)}{y^3}$$

for some cubic polynomial $h(y)$. Then

$$\int \frac{dx}{\sqrt{f(x)}} = \int \frac{\frac{-dy}{y^2}}{\sqrt{(x-\alpha)g(x)}} = \int \frac{-dy}{y^2\sqrt{(x-\alpha)g(x)}} =$$

$$\int \frac{-dy}{\sqrt{y^4\left(\frac{1}{y}\right)\left(\frac{h(y)}{y^3}\right)}} = -\int \frac{dy}{\sqrt{h(y)}}$$

as desired.

We are now left with only the degenerate cases. Most are handled by standard integration techniques, (see Exercise 1) but a few, such as

$$\int \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}$$

have been already handled in Exercise 3. The ambitious reader can chase down all these cases and verify this statement. We have thus achieved our desired goal for the section.

Exercise 7 As with Lagrange's method for finding the roots of a quartic, the integration methods above are nicer in theory than in practice. In practical examples one would use numerical integration. Try to derive, or even verify, the following:

a)

$$\int \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\mathrm{cn}^{-1}\left(\frac{x-1-\sqrt{3}}{x-1+\sqrt{3}}\right)+C, \ \kappa = \sin\frac{\pi}{12}$$

b)

$$\int \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\mathrm{cn}^{-1}\left(\frac{\sqrt{3}-1+x}{\sqrt{3}+1-x}\right)+C, \ \kappa = \sin\frac{5\pi}{12}$$

Exercise 8 There are other types of integrals that can be given in terms of the elliptic functions. Verify the following.

a)

$$\int (1+x^2)^{-\frac{3}{4}}\,dx = \sqrt{2}\mathrm{cn}^{-1}\left((1+x^2)^{-\frac{1}{4}}\right), \ \kappa = \frac{\sqrt{2}}{2}$$

b)

$$\int \kappa \mathrm{cn}u\,du = \cos^{-1}(\mathrm{dn}u)+C$$

c)

$$\int \kappa \mathrm{sn}u\,du = \cosh^{-1}\left(\frac{\mathrm{dn}u}{\breve{\kappa}}\right)+C$$

d)

$$\int \kappa \mathrm{sn}u\,du = \log\frac{\mathrm{dn}u+\kappa\mathrm{cn}u}{\breve{\kappa}}+C$$

6.6 The Complex Theory

While interesting, the previous section is considered to have little modern importance since numerical methods are obviously more convenient. But elliptic functions do play an important role in modern Mathematics because of the very nice properties they have as complex functions of a complex variable. In this section we will just outline some of this theory. It will be loosely organized around the names of the important Mathematicians who made progress in the subject.

Abel and Liouville

As mentioned earlier, Abel was the Mathematician who was largely responsible for the key ideas on elliptic functions, although if he hadn't had his key insights it is likely that Gauss would have eventually published his similar ideas. The unpleasant part of the integration of $\int f(x)^{-\frac{1}{2}}\,dx$ was the number of cases involved, we had to worry about real or complex roots of $f(x)$ and where the variables of integration were in terms of these roots. Abel was aware that if the functions sn and sn^{-1} could be treated as complex functions then many of the special cases would be the same. Abel had an idea about what these complex functions would look like and, in particular, realized that for $0 < \kappa < 1$ sn, cn and dn would be periodic not just in the real direction but also in another complex direction. Today, any doubly periodic complex function is known as an elliptic function.

Liouville worked much of the general theory of such functions, noting that they had a "period parallelogram", i.e. a parallelogram in the complex plane with vertices $0, \omega_1, \omega_3 = \omega_1 + \omega_2, \omega_2$ such that by periodicity it is enough to know the behavior of the function in this parallelogram to know the function over the entire complex plane. These functions were not defined at all points for there were isolated points known as "poles" w near which the function behaved similarly to the function $1/(z-w)^m$ (m is called the multiplicity). Liouville showed that the sum of the multiplicities of the zeros in the period parallelogram was equal to the sum of the multiplicities of the poles and that if the elliptic function was not constant then these multiplicities added to at least 2.

Note that in the Abel-Liouville sense the trigonometric functions ($\kappa = 0$) are not special cases of elliptic functions since as complex functions they are not doubly periodic, for example they become unbounded in the imaginary direction, and they have no poles.

Jacobi and Fourier

To treat the complex case Jacobi defined Theta functions (which had actually been described earlier by Fourier)

$$\Theta_1(z) = i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-\frac{1}{2})^2} e^{(2n-1)\pi i z}$$
$$\Theta_2(z) = \sum_{n=-\infty}^{\infty} q^{(n-\frac{1}{2})^2} e^{(2n-1)\pi i z}$$
$$\Theta_3(z) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi i z}$$
$$\Theta_4(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2n\pi i z}$$

where $q = e^{\pi i \tau}$ for a fixed complex number τ with $Im(\tau) > 0$. While the functions Θ_j are not periodic they are quasi-periodic and so the quotient of any two Θ_j is periodic with real period 2 and imaginary period 2τ.

In particular given $0 < \kappa < 1$, $\breve{\kappa} = \sqrt{1-\kappa^2}$,

$$K = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-\kappa^2 \sin^2 x}} \text{ and } \breve{K} = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-\breve{\kappa}^2 \sin^2 x}} \tag{6.24}$$

as in section 6.3 then

$$\mathrm{sn}(z) = \frac{1}{\sqrt{\kappa}} \frac{\Theta_1\left(\frac{z}{2K}\right)}{\Theta_4\left(\frac{z}{2K}\right)} \tag{6.25}$$
$$\mathrm{cn}(z) = \frac{\sqrt{\breve{\kappa}}}{\sqrt{\kappa}} \frac{\Theta_2\left(\frac{z}{2K}\right)}{\Theta_4\left(\frac{z}{2K}\right)} \tag{6.26}$$
$$\mathrm{dn}(z) = \sqrt{\breve{\kappa}} \frac{\Theta_3\left(\frac{z}{2K}\right)}{\Theta_4\left(\frac{z}{2K}\right)} \tag{6.27}$$

The Jacobi elliptic functions are now seen to be elliptic in the sense of Abel and Liouville. The vital statistics are given by the chart:

function	real period	imaginary period	zeros	poles
$\mathrm{sn}(z)$	$4K$	$2i\breve{K}$	$0, 2K$	$i\breve{K}, 2K + i\breve{K}$
$\mathrm{cn}(z)$	$4K$	$2K + 2i\breve{K}$	$K, 3K$	$i\breve{K}, 4K + i\breve{K}$
$\mathrm{dn}(z)$	$2K$	$4i\breve{K}$	$K + i\breve{K}$	$i\breve{K}, 3i\breve{K}$

The zeros and poles are only given for one period parallelogram, to get other points where these functions have zeros or poles simply add multiples of the real and/or imaginary periods. Note that all the zeros and poles are simple, i.e. have multiplicity 1.

One application of the Θ-function approach to the Jacobi elliptic functions is that we can get some good approximations to these functions almost without resorting to numerical methods. For example it can be shown that if, say, $0 < \kappa < .98$ then q is small enough to make the series converge quickly so the following are reasonable approximations, at least for the purpose of graphing:

$$\mathrm{sn}(z) \approx \frac{1}{\sqrt{\kappa}} \frac{2q^{\frac{1}{4}} \sin \frac{\pi z}{2K} - 2q^{\frac{9}{4}} \sin \frac{3\pi z}{2K}}{1 - 2q \cos \frac{\pi z}{K}} \tag{6.28}$$

$$\mathrm{cn}(z) \approx \frac{\sqrt{\breve{\kappa}}}{\sqrt{\kappa}} \frac{2q^{\frac{1}{4}} \cos \frac{\pi z}{2K} + 2q^{\frac{9}{4}} \cos \frac{3\pi z}{2K}}{1 - 2q \cos \frac{\pi z}{K}} \tag{6.29}$$

$$\mathrm{dn}(z) \approx \sqrt{\breve{\kappa}} \frac{1 + 2q \cos \frac{\pi z}{K}}{1 - 2q \cos \frac{\pi z}{K}} \tag{6.30}$$

Here $\kappa, \breve{\kappa}, K, \breve{K}, \tau = i\frac{\breve{K}}{K}$, and $q = e^{\pi i \tau}$ are as above and $K, \breve{K}$ still must be calculated numerically.

These days real periodic functions are usually analyzed by Fourier series. There are various formulations of these. Since our Jacobi Elliptic functions have different periods for different κ we want to normalize to period 2 by multiplying the variable by half the period. In other words, if $g(x) = g(x + 2p)$ for all x then a Fourier series for g is

$$g(px) = \sum_{n=0}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$$

Most periodic functions, even slightly discontinuous ones, have Fourier series that converge to the function at most points. An advantage of studying the Jacobi Elliptic Functions is that you now have some non-trivial examples to apply the Fourier series to! Fourier series for the Jacobi Elliptic functions (where $\kappa, \breve{\kappa}, K$, and q are as above) are given by:

$$\operatorname{sn}(2Kx) = \frac{2\pi}{K\kappa} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}} \sin(2n+1)\pi x \tag{6.31}$$

$$\operatorname{cn}(2Kx) = \frac{2\pi}{K\kappa} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}} \cos(2n+1)\pi x \tag{6.32}$$

$$\operatorname{dn}(2Kx) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=0}^{\infty} \frac{q^{n}}{1+q^{2n}} \cos 2n\pi x \tag{6.33}$$

Weierstrass

Weierstrass was perhaps partially motivated by wanting to simplify the integration techiques for $\int f(x)^{-\frac{1}{2}}\,dx$ discussed in the previous section by eliminating all the special cases. He knew that if $f(x)$ was a biquadratic the problem could be reduced to the cubic case. Moreover, linear changes of variables are not interesting, so as in the Cardano-Viete solution of the cubic in Section 4.4, one can perform a linear change of variables to eliminate the x^2 term of the cubic. Weierstrass' form is slightly different: his typical cubic is $g(x) = 4x^3 - g_2x - g_3$.

Weierstrass then was able to find an elliptic (in the sense of Abel and Liouville) function $\wp(z)$ (depending on parameters g_2, g_3) such that an inverse function (with appropriate range) would satisfy

$$\wp^{-1}(x) = \int_x^{\infty} \frac{du}{\sqrt{4u^3 - g_2u - g_3}} \tag{6.34}$$

Using our integration formulas of the previous section we can actually calculate $\wp^{-1}(x)$ in one case, mainly where $g(x)$ has 3 distinct real roots. So suppose $g(x) = 4(x-\alpha)(x-\beta)(x-\gamma)$ where $\alpha > \beta > \gamma$. As in Exercise 6 let

$$\kappa_0 = \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}} \quad \text{so} \quad \breve{\kappa}_0 = \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}$$

and let $K, \breve{K}$ be defined as in equation (6.24) for $\kappa = \kappa_0$.

When $x > \alpha$ then

$$\wp^{-1}(x) = \int_x^\infty \frac{du}{\sqrt{g(u)}} = \int_x^\infty \frac{du}{2\sqrt{(u-\alpha)(u-\beta)(u-\gamma)}} =$$

$$\lim_{u\to\infty} \frac{1}{\sqrt{\alpha-\gamma}}\mathrm{sn}^{-1}\left(\sqrt{\frac{u-\alpha}{u-\beta}}\right) - \frac{1}{\sqrt{\alpha-\gamma}}\mathrm{sn}^{-1}\left(\sqrt{\frac{x-\alpha}{x-\beta}}\right) =$$

$$\frac{K}{\sqrt{\alpha-\gamma}} - \frac{1}{\sqrt{\alpha-\gamma}}\mathrm{sn}^{-1}\left(\sqrt{\frac{x-\alpha}{x-\beta}}\right)$$

since $\lim_{u\to\infty} \frac{u-\alpha}{u-\beta} = 1$ and $\mathrm{sn}^{-1}(1) = K$ since $\kappa = \kappa_0$. Weierstrass denotes $\omega_1/2 = \frac{K}{\sqrt{\alpha-\gamma}}$ which will be the real half period of $\wp$. Thus we note from this formula that as we start from $x = \infty$ and go down to $x = \alpha$ then $\wp^{-1}$ monotonically takes real values from 0 to $\omega_1/2$.

When $\alpha > x > \beta$ then $g(x)$ is negative but Weierstrass is allowing complex values so $\sqrt{g(x)} = \pm i\sqrt{-g(x)}$ with the choice of sign somewhat arbitrary. But Weierstrass wants us to chose the positive so that $\frac{1}{\sqrt{g(x)}} = \frac{-i}{\sqrt{-g(x)}}$. Then

$$\int_x^\infty \frac{du}{\sqrt{g(u)}} = \int_x^\alpha \frac{du}{\sqrt{g(u)}} + \int_\alpha^\infty \frac{du}{\sqrt{g(u)}} = (-i)\int_x^\alpha \frac{du}{\sqrt{-g(u)}} + \frac{\omega_1}{2} =$$

$$\frac{\omega_1}{2} - \frac{-i}{\sqrt{\alpha-\gamma}}\left[\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-x}{\alpha-\beta}}\right) - \mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\alpha}{\alpha-\beta}}\right)\right] =$$

$$\frac{\omega_1}{2} + \frac{i}{\sqrt{\alpha-\beta}}\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-x}{\alpha-\beta}}\right)$$

Here $\kappa = \breve{\kappa}_0$ so when $x = \beta$ in the last term we get

$$\frac{i}{\sqrt{\alpha-\beta}}\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\beta}{\alpha-\beta}}\right) = \frac{i}{\sqrt{\alpha-\beta}}\mathrm{sn}^{-1}(1) = \frac{i\breve{K}}{\sqrt{\alpha-\beta}}$$

which Weierstrass calls $\omega_2/2$. Thus we see that as x decreases from α to β then $\wp^{-1}(x)$ travels parallel to the y-axis from $\omega_1/2$ to $\omega_1/2 + \omega_2/2 = \omega_3/2$

We will let the reader apply the last two formulas of Exercise 6 in a similar fashion to show that as x goes down from β to γ then $\wp^{-1}(x)$ goes in the positive real direction to $\omega_1 + \omega_2/2$. And finally, as x decreases from γ to $-\infty$ then $\wp^{-1}$ moves upwards in the imaginary direction to $\omega_1 + \omega_2 = \omega_3$

Incidentally, we have shown that

$$\int_{-\infty}^\infty \frac{dx}{\sqrt{g(x)}} = \omega_3$$

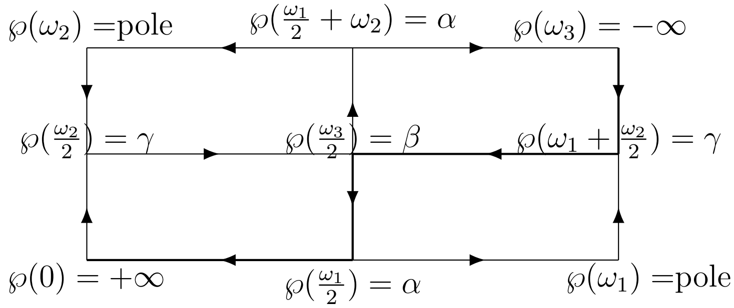

Figure 6.4: The period parallelogram for $\wp$

but because we have made choices it is not clear what this actually means.

But we can now get a good idea of how $\wp$ works by inverting our calculation of $\wp^{-1}$. For one thing, we have shown that $\wp(\omega_1/2) = \alpha, \wp(\omega_3/2) = \beta$ and $\wp(\omega_2/2) = \gamma$. Further $\wp(0)$ and $\wp(\omega_3)$ are various flavors of infinity which really means that $\wp$ has poles at those points. (It is generally understood that there is just one value of infinity for the complex plane, anyway.) Also we see that $\wp$ takes monotonically decreasing real values on the piecewise linear path from ω_3 to $\omega_1 + \omega_2/2$ to $\omega_3/2$ to $\omega_1/2$ to 0, and on that route $\wp(z)$ takes on every real value exactly once.

If we add to this the facts that $\wp$ is periodic with periods ω_1 and ω_2 and that $\wp$ is even (i.e. $\wp(z) = \wp(-z)$ for all complex z), we must take Weierstrass' word on this, we can fill in more information on the period parallelogram. For example, to see how $\wp$ behaves on the interval from $\omega_2/2$ to $\omega_3/2$ we can take the interval from $\omega_3/2$ to $\omega_1 + \omega_2/2$ and subtract ω_3 which is the sum of the two periods. We then get by periodicity that on the interval from $-\omega_3/2$ to $-\omega_2/2$ $\wp$ behaves the same as on the former interval. But by the even property we get the behavior on the interval we wanted, mainly $\wp(z)$ is real and increasing from γ to β on the interval from $\omega_2/2$ to $\omega_3/2$.

Figure 6.4 now summarizes what we now know about $\wp$ on the period parallelogram. On the indicated lines $\wp(z)$ is real and the arrows give the direction in which $\wp(z)$ is increasing. Note the heavy line is the path on which we implicitly calculated $\wp(z)$ by integration. We have poles at the corners and, from the considerations of Section 4.8, since $g(x)$ has coefficient 0 for the quadratic term and hence $\alpha + \beta + \gamma = 0$, we conclude that we have one zero that lies either on the segment from $\omega_1/2$ to $\omega_3/2$ or on the segment from $\omega_3/2$ to $\omega_1 + \omega_2/2$. Since those two segments are mirrored by the other two interior segments we must have one

Figure 6.5: How to make a Torus

addional zero on one of them, unless it happens that $\beta = 0$ in which case we have a zero of multiplicity 2. On the rectangle, $\wp$ takes on the values α, γ at two points but $\wp(z) = \beta$ only at $z = \omega_3/2$. Finally, although we haven't proved it, $\wp(z)$ takes imaginary values on the interior of the rectangles.

Weierstrass now does something that you may find strange. He cuts out the period parallelogram and rolls it into a cylinder by gluing the segment 0 to ω_1 to the segment ω_2 to ω_3. Then he curves this around so he can glue segment 0 to ω_2 to the segment ω_1 to ω_3. He obtains a torus. See Figure 6.5.

Because of the double periodicity the points Weierstrass has glued together take the same value under $\wp$ so Weierstrass has a well defined function from the torus to the complex plane. Since the point $\omega_1/2$ has been glued to $\omega_1/2 + \omega_2$ now $\wp(z) = \alpha$ for only one point, similar to the case for β. The same applies also to γ. With the other gluing we find that each of the other real values is attained twice (not 4 times).

Weierstrass has more up his sleeve (perhaps literally, the history books tell us that he often wrote on his sleeves when he had an urgent idea that couldn't wait for him to find paper). The derivative $\wp'(z)$ is also doubly periodic (this should be obvious). Writing $z = \wp^{-1}(x)$ (or $\wp(z) = x$) we find that

$$\frac{dz}{dx} = \frac{d}{dz}\int_x^\infty \frac{du}{\sqrt{g(u)}} = \frac{-1}{\sqrt{g(x)}}$$

by the Fundamental Theorem of Calculus. Taking reciprocals we have

$$\wp'(z) = \frac{d\wp}{dz} = -\sqrt{g(x)}$$

Thus letting $y = \wp'(z)$ we have the equation $y^2 = g(x)$. Of course we have only shown that this works on our path, but in fact this works everywhere on the period parallelogram (or torus!) as long as we take $x = \wp(z)$. Weierstrass now has a function $z \to (x, y) = (\wp(z), \wp'(z))$ from the torus to the cartesian product of the complex plane with itself (more precisely, to the complex projective plane). One technicality is that he must choose which of the two square roots of $g(x)$ to take in order to have y well defined. Fortunately he can do this and in fact in such a way that each choice is made exactly once. That is possible since $\wp$ attains each x value twice except for $x = \alpha, \beta$ or γ and there since $g(x) = 0$ no choice need be made! *What Weierstrass has done is that he has described a 1-1 correspondance between the points on the torus and the complex solution set of the equation* $y^2 = g(x)$. We have only outlined this in the case when $g(x)$ has three distinct real solutions but in fact it works for any $g(x)$ (even with complex g_2, g_3) as long as the roots are distinct. Even this is easy to test as this requires, by Section 4.8, the discriminant $\Delta = 4(g_2^3 - 27g_3^2) \neq 0$

Thus Weierstrass set out to calculate some integrals but on the way discovered geometrically what the complex solution set of the algebraic equation $y^2 = 4x^3 - g_2x - g_3$ looks like. It looks like a torus. Because this solution set is parameterized by elliptic functions, the algebraic curves

$$y^2 = 4x^3 - g_2x - g_3, \quad g_2^3 - 27g_3^2 \neq 0$$

have since been known as "elliptic curves" even though they don't look anything like ellipses.

Riemann and later

This changed everything. Mathematicians no longer thought of elliptic functions in terms of integrals but in terms of their applications to geometry. To be completely historically correct the credit does not go to Weierstrass alone, even Abel had to some extent anticipated this development, and though more difficult, the same result can be derived using Θ-functions. But certainly Weierstrass had given the most elegant formulation.

Riemann took the next step by his invention of Riemann Surfaces. He was able to deal with curves of higher degree and also higher dimensional geometry. His work, in large part motivated by the recent results on elliptic functions and curves, led to the formation of the fields of differential geometry and topology. After Weierstrass pointed out a gap in some of Riemann's work resulting from some technical problems in analysis, Clebsch reworked some of Riemann's results starting from the point of view of the curve and using abstract algebra rather than geometry. This work, carried on by his pupil Max Noether, led to the modern formulation of the field of algebraic geometry.

But now elliptic curves and functions, freed from analysis, could be applied to number theory. This is where most of the modern interest in these ideas lies. In fact, Andrew Wiles recent proof the Fermat Conjecture was simply a sidlight of his studies of elliptic curves over number fields. Other areas in which elliptic curves have been recently used is in factoring methods and encryption methods for secure internet transmissions.

Elliptic curves and functions are considered some of the most advanced and difficult areas of mathematics today. New progress is being made. But it all started with some mundane integration problems.