Theory of Equations

Lesson 9

by

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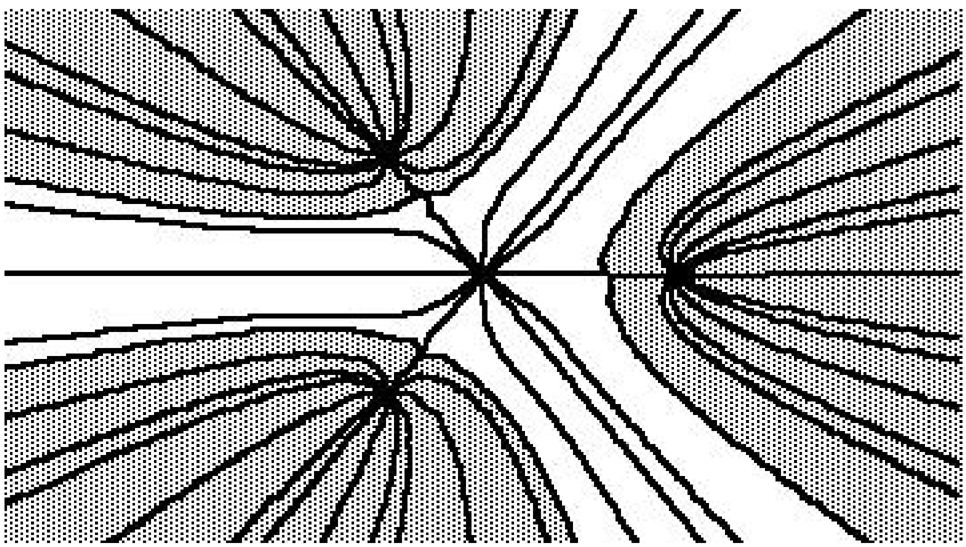


Chapter 4

Ancient and Modern Algebra

THE EXACT SOLUTION OF POLYNOMIAL EQUATIONS

In this chapter we turn our attention to exact solutions of polynomial equations. The Babylonians used tables of values of $n^3 + n$ to find numerical solutions of easy cubic equations, however the Greeks around the time of Euclid became obsessed with exactness. Largely due to the dominant influence of Euclid's Elements there was a great interest in exact solutions up to the proof by Abel and Galois of the impossibility of such solutions in general. In the first few sections of this chapter we review some of the history of exact solutions. A good overview of this history is included in B.L van der Waerden's *A History of Algebra*.

4.1 Solutions of Quadratic Equations

Although Euclid, in keeping with the philosophy of his time, rejected all numbers he still indirectly considered quadratic equations. His version of the solution of the quadratic equation $ax + x^2 = b^2$ is the geometric theorem:

> If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole (with the added straight line) and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

Essentially this is completing the square.

The Arab mathematician Al Khwarizmi in 825 reacted negatively to the Euclidean avoidance of numbers and invented algebra as an alternative to geometry. Unfortunately slightly later Arabs, feeling uneasy about the lack of preciseness of the algebra brought back in the geometry to justify the algebraic arguments. Al Khwarizmi's algebra reached Europe in this form. Thus we illustrate Arabic methods from the work of Abu Kamil (see for instance Martin Levey, *The Algebra of Abu Kamil*), who wrote about 75 years after Al-Khwarizmi. Abu Kamil starts out his algebra with the statement:

> First it is necessary for the reader of this book to know that there are three categories according to Muhammad al-Khwarizmi in his book. They are roots, squares and numbers.

Thus in the equation $x^2 + 10x = 39$, x is the root, x^2 is the square and 39 is the number. This equation is then phrased as "the square plus 10 roots equal 39." The solution which Abul Kamil gives is completing the square.

> You always take 1/2 the roots; in this problem it is 5. You multiply it by itself; it is 25. One adds this to 39; it is 64. Take its root; it is 8. Subtract from it 1/2 the roots or 5; 3 remains. It is the root of the square. The square is 9.

It should be noted that the desired solution is the square, not the root. Abul Kamil then justifies this solution method with geometry (see Figure 4.1).

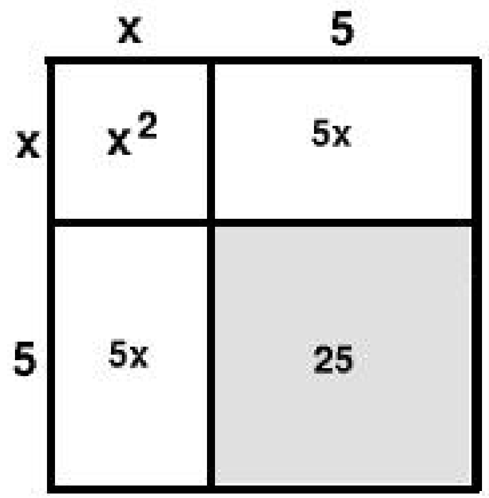
Abul Kamil goes on to do 68 more problems involving quadratic equations or equations leading to quadratics.

This philosophy on solution of equations, and especially the idea that "the root" was a length, and the and "the square" was an area survived through the 16th century along with the rhetorical notation. Modern algebraic notation was introduced by Stevin and Harriot in the early 17th century. Descartes was instrumental popularizing this new notation and in separating the idea of "degree" from the geometrical notion of dimension. Although a bit out of sequence historically we give Descartes' solution of the quadratic by way of contrast to that of Abu Kamil.

Descartes says:

> For example, if I have $z^2 = az + bb$, I construct a right triangle $\triangle NLM$ with one side LM, equal to b , the square root of the known quantity bb, and the other side, LN, equal to $\frac{1}{2}a$, that is, to half the other known quantity which was multiplied by z, which I supposed to be the unknown line. Then prolonging MN, the hypotenuse of this triangle to O, so that NO is equal

Figure 4.1: Abu Kamil's solution to $x^2 + 10x = 39$



to NL, the whole line OM is the required line z. This is expressed in the following way:

$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$$

This is essentially the first statement in more or less modern language of the general quadratic equation. (See Figure 4.2).

Incidentally, Descartes justification of his formula is algebraic, the point of his geometry is to establish that the number $\sqrt{aa/4 + bb}$ exists, i.e. it is the length of the hypotenuse of the triangle $\triangle LMN$.

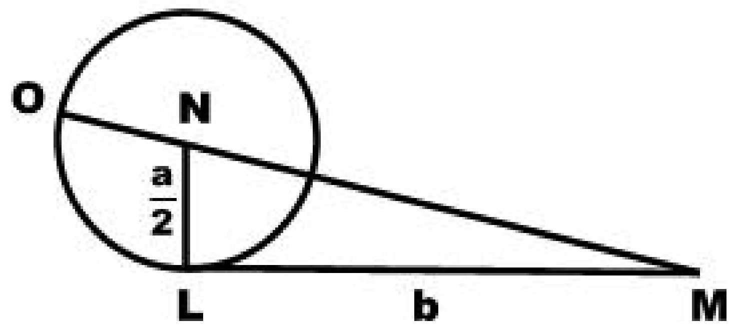
Although later in his book, Descartes allows negative coefficients and negative roots, at this point in his book, the first chapter on analytic geometry, only positve coefficients and roots are considered.

We now know that the complete solution to all quadratic equations $ax^2 + bx + c = 0$ is given by the general quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This solution is valid even when a, b and c are complex numbers and when the quantity inside the square root sign is positive, negative or imaginary.

Figure 4.2: Descartes' Solution of $z^2 = az + b^2$

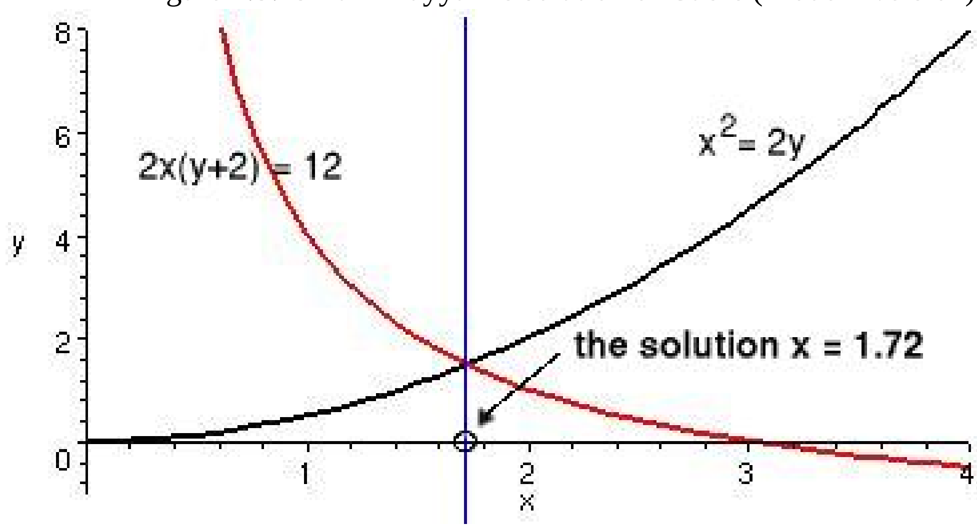


4.2 Omar Khayyam and Viete

SOLUTION OF THE CUBIC

Omar Khayyam, the Persian poet and mathematician of the second half of the 11th century, was one of the first mathematicans to consider the cubic equation. Like Descartes 6 centuries later he avoided associating x^2 with area and x^3 with volume, but unlike Descartes who disassociated the two concepts, Omar Khayyam got around this problem with the statement "Every time we shall say in this book 'a number is equal to a rectangle', we shall understand by the 'number' a rectangle of which one side is unity, and the other a line equal in measure to the given number..."

Figure 4.3: Omar Khayyam's solution of Cubic (Modern version)



Omar Khayyam then solves the equation such as “a cube and 4 sides equals 12”, i.e. $x^3+4x = 12$ by making making the substitution $x^2 = 2y$, and noting that $x^2 = 2y$ is the equation of a parabola. The cubic is then written $2xy+4x = 12$ or $2x(y+2) = 12$ which Khayyam notices is the equation of a hyperbola which intersects the parabola in one point. The x coordinate of this point is the solution, which is described geometrically, but not algebraically. Other cubic equations were solved by other use of various conics. Of course Omar Khayyam did not use the modern algebraic notation used above, rather he used ratios, and of course, he was not able to graph using cartesian coordinates which were not invented until 350 years later.

Again, jumping out of historical sequence, we mention Viete’s method of finding the real roots of a cubic. Viete was a French mathematician who made advances in algebra, but is most noted for his many trigonometric identities. He developed an algebraic notation which was an advance over what had been used previously and which was adopted by many French 17th century mathematicians such as Fermat. It is fortunate for us that neither Descartes or Leibniz adopted Viete’s notation, for this notation is inferior to our modern notation.

Viete first uses a change of variables to reduce a cubic to the form $y^3 + py = q$, see Theorem 2.5.1, this is a reduction we now use in all exact solutions of cubics. Viete then lets $h = \sqrt{4|p|/3},\ k = 3/(h|p|)$ and substitutes $y = hz$ and then multiplies through by k. One gets either the equation

$$4z^3 + 3z = C$$

or the equation

$$4z^3 - 3z = C.$$

In the case that the original equation had three real roots the second equation is obtained with $|C| < 1$. Viete then used his trigonometric identity $\cos 3\phi = 4\cos^3\phi - 3\cos\phi$ and the substitution $z = \cos\phi$ to obtain the equation $\cos 3\phi = C$, of which he notes there are three roots. If $|C| \geq 1$ or the first equation is obtained it is not clear that Viete knew how to proceed, however it is now known that similar substitutions involving the hyperbolic $\sinh$ and $\cosh$ will provide the one real root (see *Birkhoff and MacLane*). It should be noted that Viete realized from his method that solving cubic equations with three real roots was equivalent to the problem of trisecting an angle.

4.3 History of the Cubic and Biquadratic

The solution of the cubic equation by algebraic methods was first accomplished by Scipione del Ferro, Professor of Mathematics in Bologna, Italy, apparently in the year

1515. del Ferro did not publish his solution but it was known to some of his pupils, notably Antonio Maria Fiore, who in 1535 challenged a young man nicknamed Tartaglia to a contest in solving the equation "the first power plus the cube equal to a number." Tartaglia managed to solve the problem the night before the contest and defeated Fiore.

The distinguished physician Cardano, who was at that time engaged in writing an elementary book on mathematics, heard about Tartaglia's success and in 1539 tried to persuade Tartaglia to expain his method. Tartaglia had his own intentions to publish and first refused but later was induced to give his method to Cardano, but even then only in obscure verses, and only after Cardano swore a solemn oath not to reveal the method until Tartaglia had published.

Cardano mastered and expanded Tartaglia's method and became impatient to publish. In 1543 Cardano received permission to inspect del Ferro's papers and found the original solution at which point he published his famous *Ars Magna* (it appeared in 1545). Tartaglia was furious and the ensuing debate between Tartaglia, Cardano and Cardano's former houseboy and student Ferrari lasted over 30 years. Ferrari, who had been present when Tartaglia gave Cardano the method, discovered how to solve the biquadratic (fourth degree) using the solution to the cubic. Cardano included this in his book however with the qualification:

> Although a long series of rules might be added and a long discourse given about them, we concude our detailed consideration with the cubic, others being merely mentioned, even if generally, in passing. For as the first power refers to a line, the square to a surface, and the cube to a solid body, it would be very foolish to go beyond this point. Nature does not permit it.

Cardano used a rhetorical notation (no symbols) and did not allow negative coefficients. Thus his book has separate chapters for the 13 different cases of cubics that Cardano identifies. A nice translation of *Ars Magna* has been given by T.R. Witmer and published under the title *The Great Art*.

The one case where the cubic has three real roots was called the irreducible case. Cardano could not solve this case since the real roots could be found only by taking the square roots of imaginary numbers. Rafael Bombelli in 1572 mastered enough complex arithmetic to solve this problem. Viete, who did not accept negative numbers, much less imaginary ones, invented his trigonometric method to handle this case.

4.4 Algebraic solution of the Cubic

The modern algebraic solution of the cubic is essentially that described by Cardano with some improvements by Viete. However, since we can now take square and cube roots of negative and imaginary numbers we need only one case to handle all cubics, including the "irreducible case." We give an exposition of the method combining that of *Birkhoff and MacLane* and *Uspensky*.

Given a cubic polynomial equation (we can assume monic)

$$x^3 + ax^2 + bx + c = 0 \tag{4.1}$$

the first step is to do a change of variables to eliminate the x^2 term. This can be done using Horner's process, see theorem 2.5.1. In particular, letting $y = x + a/3$ we get the equation

$$y^3 + py + q = 0 \tag{4.2}$$

where $p = f'(-a/3) = b - a^2/3$ and $q = f(-a/3) = c - ba/3 + 2a^3/27$. If we can solve Equation 4.2 for y then the solutions for x are given by $x = y - a/3$. Thus for the rest of this section we will assume the equation to be solved is Equation 4.2. The trick (due to Viete) is to make the substitution in Equation 4.2 of

$$y = w - \frac{p}{3w} \tag{4.3}$$

From the binomial theorem and algebra we get

$$w^3 - \frac{p^3}{27w^3} + q = 0$$

or

$$(w^3)^2 + qw^3 - p^3/27 = 0 \tag{4.4}$$

which is quadratic in w^3 and can be solved by the general quadratic equation as

$$w^3 = -q/2 \pm \sqrt{q^2/4 + p^3/27}$$

Thus we have two possible values for w^3, and, unless one of these is 0, three possible cube roots for value of w^3. However, upon substituting in Equation 3 we find only 3 separate values of y i.e. the three cube roots of just one of the solutions of Equation 4.4. In fact, we only need to calculate one cube root.

To see this, it is known that the two roots α, β of the quadratic equation $x^2+bx+c = (x-\alpha)(x-\beta) = 0$ satisfy $\alpha\beta = c$. Thus the two roots (i.e. values of w^3) of Equation

4.4 multiply to give $-p^3/27$. Thus if A is a cube root of one solution of Equation 4.4, i.e. A^3 is a solution to

$$x^2 + qx - p^3/27 = 0 \tag{4.5}$$

we can let $B = -p/(3A)$ and it is seen that B^3 is the other solution to Eq. 4.5 since $A^3B^3 = -p^3/27$. A is one solution for w in Equation 4.4 so $y = w - p/(3w) = A - p/(3A) = A + B$ is a solution for Eq. 4.2. B is another solution for w but then $-p/(3B) = A$ since $AB = -p/3$ so the solution obtained for Equation 2 is again $B + A$.

Moreover, if $\gamma = -1/2 + \sqrt{3}/2i$ then γ is a cube root of 1 and if A is the cube root of some complex number then $\gamma A, \gamma^2 A$ are the others. Thus if A^3 satisfies

$$A^3 = -q/2 + \sqrt{q^2/4 + p^3/27}$$

and $B = -p/(3A)$, the three solutions to Equation 4.2 are

$$\begin{aligned} y_1 &= A + B \\ y_2 &= \gamma A + \gamma^2 B \\ y_3 &= \gamma^2 A + \gamma B \end{aligned}$$

These last three equations are known as Cardano's equations.

As an example we consider the equation

$$y^3 - 15y - 4 = 0$$

solved by Bombelli in 1572. Then $-q/2 + \sqrt{q^2/4 + p^3/27} = 2 + \sqrt{16/4 - 15^3/27} = 2 + \sqrt{4 - 125} = 2 + \sqrt{-121} = 2 + 11i$. Finding a cube root of $2 + 11i$ by, say, DeMoivre's law (Bombelli most likely used trial and error) we get $A = 2 + i$. Thus $B = -p/(3A) = 5/(2 + i) = 2 - i$ and the three solutions are

$$\begin{aligned} y_1 &= A + B = [2 + i] + [2 - i] = 4 \\ y_2 &= \gamma A + \gamma^2 B \\ &= [(-1 - \sqrt{3}/2) + (\sqrt{3} - 1/2)i] + [(-1 - \sqrt{3}/2) - (\sqrt{3} - 1/2)i] \\ &= -2 - \sqrt{3}. \\ y_3 &= \gamma^2 A + \gamma B \\ &= [(-1 + \sqrt{3}/2) + (-\sqrt{3} - 1/2)i] + [(-1 + \sqrt{3})/2) - (-\sqrt{3} - 1/2)i] \\ &= -2 + \sqrt{3} \end{aligned}$$

We summarize Cardano's solution as follows, while not quite as compact as the quadratic equation, it is just as mechanical.

To solve $x^3 + ax^2 + bx + c = 0$

Let $p = b - a^2/3, q = c - \frac{ba}{3} + \frac{2a^3}{27}$ Note that if $a = 0$ then $p = b$ and $q = c$.

Let A be any cube root of $-q/2 \pm \sqrt{q^2/4 + p^3/27}$, where the sign $\pm$ is chosen so that $A \neq 0$. Let $B = -p/(3A)$ and $\gamma = \frac{-1+\sqrt{3}i}{2}$.

Then the solutions are

$$x_1 = A + B - \frac{a}{3}$$
$$x_2 = \gamma A + \gamma^2 B - \frac{a}{3}$$
$$x_3 = \gamma^2 A + \gamma B - \frac{a}{3}$$

Maple Implementation

To obtain Cardano's solutions for a cubic f simply use the command

```
solve(f, x);
```

But unless Maple sees an obvious simplification the solution will be quite literally the one above.

In the following two exercises do all work by hand, do not use Maple, except, perhaps, as a check.

Exercise 4.4.1 [30 points] Use the method of this section to find the 3 complex roots of $x^3 + 6x - 20 = 0$. Give an exact answer.

Exercise 4.4.2 [40 points] An open wooden box (without a top) is in the shape of a cube with each outer edge 10 inches long. If the volume is 500 cubic inches what is the thickness of the wood (assuming uniform thickness). Give an exact, not decimal, answer! This problem is taken from *Uspensky*.

We remark that the number inside the radical sign is $(-4p^3 - 27q^2)/(-108)$ and the numerator has a special meaning. Suppose that Equation 4.2 has the three complex roots z_1, z_2, and z_3. Then $(y - z_1)(y - z_2)(y - z_3) = y^3 + py + q$ and so we get

$$\begin{aligned} z_1 + z_2 + z_3 &= 0 \\ z_1z_2 + z_1z_3 + z_2z_3 &= p \\ -z_1z_2z_3 &= -q \end{aligned}$$

Now let $D = (z_1 - z_2)^2(z_1 - z_3)^2(z_2 - z_3)^2$. A straight forward multiplication using the three formulas above shows that $D = -4p^3 - 27q^2$, i.e. the numerator under the radical sign. D is called the *discriminant* of Equation 4.2. This should be compared with the quadratic case where if z_1, z_2 are the two roots of $x^2 + bx + c = 0$ then $(z_1 - z_2)^2 = b^2 - 4c$. As in the quadratic case, the discriminant in the cubic case gives us information on the number of imaginary roots:

Theorem 4.4.1 *Let $D = -4p^3 - 27q^2$ be the discriminant of the equation $y^3 + py + q = 0$ where p and q are real. If $D > 0$ this equation has 3 distinct real roots, if $D < 0$ this equation has one real root and 2 imaginary roots. If $D = 0$ this equation has multiple real roots.*

Note that in the case of 3 real roots that $q^2/4 + p^3/27 < 0$ so A, B in the solution are imaginary numbers. So imaginary numbers may be required to find the real solutions of a real equation.

4.5 Solution of the Biquadratic Equation

The method still in use is essentially the method due to Ferrari. The first step, as in the case of the cubic is to simplify the equation, by a linear change of variable using 2.5.1 to get rid of the x^3 term. Thus we may assume our equation is

$$y^4 + py^2 + qy + r = 0 \tag{4.6}$$

The trick is to add $uy^2 + u^2/4$ to both sides of the equation where u is, for now, unknown. Thus we get

$$y^4 + uy^2 + u^2/4 = (u - p)y^2 - qy + (u^2/4 - r)$$

or

$$(y^2 + u/2)^2 = (u - p)y^2 - qy + (u^2/4 - r) \tag{4.7}$$

The game is to now choose u so that the quadratic on the right side is a complete square, i.e. its discriminant should be 0. Thus we should have

$$q^2 - 4(u-p)(u^2/4 - r) = 0$$

or

$$u^3 - pu^2 - 4ru + (4rp - q^2) = 0 \tag{4.8}$$

Now Equation 4.8 is a cubic in u, so we can solve (in principle!) using the methods of the previous section. It is enough to find any one root u. Equation 4.7 then becomes

$$(y^2 + u/2)^2 = (ey + f)^2$$

for

$$e = \sqrt{u-p} \text{ and } f = \sqrt{\frac{u^2}{4} - 4} = -q/2e.$$

We then have

$$(y^2 + u/2)^2 - (ey + f)^2 = 0$$

or

$$[(y^2 + u/2) + (ey + f)][(y^2 + u/2) - (ey + f)] = 0. \tag{4.9}$$

Thus solving Equation 4.6 is reduced to solving the 2 quadratics

$$\begin{aligned} y^2 + ey + (f + u/2) &= 0 \\ y^2 - ey - (f - u/2) &= 0 \end{aligned}$$

In fact, the left hand side of Eq. 4.8 evaluated at $u = p$ takes the value $-q^2$. Thus if p, q, r are all real, Theorem 2.6.3 says that there is at least one real u solving Eq. 4.8 with $u > p$. Taking this solution, e, f are both real and so the left hand side of Eq. 4.9 is simply the factorization of $y^4 + py^2 + qy + r$ into real quadratics as guaranteed by D'Alembert's Theorem, Theorem 1.9.5!

Exercise 4.5.1 [40 points] Solve $x^4 + 5x^2 + 2x + 8 = 0$ using the method of this section. Find an exact solution by hand and show work.

Maple Implementation

As before you can try

```
solve(f,x);
```

However, unless there is a major simplification, Maple will not give Ferrari's solution, often returning only `RootOf(f,x)`. Generally Maple feels that the Ferrari solution is so complicated that you don't want to see it. If you insist on seeing the full solution, give the command

```
_EnvExplicit := true;
```

before executing `solve`. You'll be sorry you did.