Theory of Equations

Lesson 8

by

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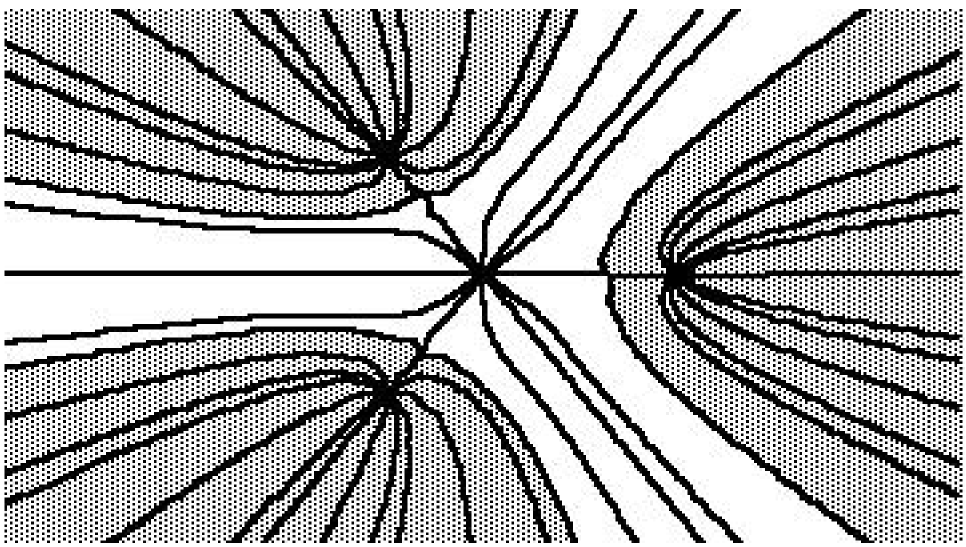


3.5 Another Curve Proof

Our last proof is again inspired by Gauss, and contains elements of both his first and fourth proofs. While this proof is not as simple or elegant as the previous proofs it does illustrate the geometry of complex polynomial functions and has some surprising connections with Newton's method which we will explore in the next section. The details of this proof can be understood with a knowledge of advanced calculus and a small amount of complex analysis. A complete proof has been provided by James D. Moor in his 1982 Masters Thesis at Northeastern Illinios University. In fact, Mr. Moor's Thesis on the FTA has been the inspiration for the first part of this chapter.

We start by generalizing the idea of the argument of a complex number. Given a complex number $z = x+yi$ with $x \neq 0$ note that there is a unique angle $\phi, -\pi/2 < \phi < \pi/2$ such that $\tan\phi = y/x$, i.e. $\phi = \arctan(y/x)$. Then from $\tan\phi = \sin\phi/\cos\phi = y/x$ we get

$$x * \sin\phi - y * \cos\phi = 0$$

Conversely, every number $z = x + yi$ which satisfies this equation either is 0 or has argument $\phi + k\pi$ for some integer k. Moreover, all purely imaginary numbers yi satisfy this equation for $\phi = \pi/2$. Thus every complex number satisfies an equation of this type for at least one $\phi, -\pi/2 < \phi \leq \pi/2$.

In this proof we consider a complex polynomial $f(z) = u(z) + v(z)i$ with real and imaginary parts $u(z)$ and $v(z)$ respectively. Fix ϕ, $-\pi/2 < \phi \leq \pi/2$, and let $\Gamma_\phi(f) = \Gamma_\phi = \{z \in \boldsymbol{C} | u(z)\sin\phi - v(z)\cos\phi = 0\}$. Thus Γ_ϕ is the set of all points z in the complex plane such that the argument of $f(z)$ is $\phi + k\pi$ for some integer k or $f(z) = 0$. As remarked above, every point in the complex plane is contained in some Γ_ϕ. In Gauss's fourth proof the "seashore" was the set $\Gamma_{\pi/2}$, and in Gauss's first proof he considered both the curves Γ_0 and $\Gamma_{\pi/2}$.

A set D of complex numbers is called *connected* if there do not exist non-empty open sets U, V of $\boldsymbol{C}$ such that $U \cap V = \emptyset$ but D is contained in the union of U and V. If a set D is not connected it is the union of (possibly infinitely many) largest possible connected subsets called "components".

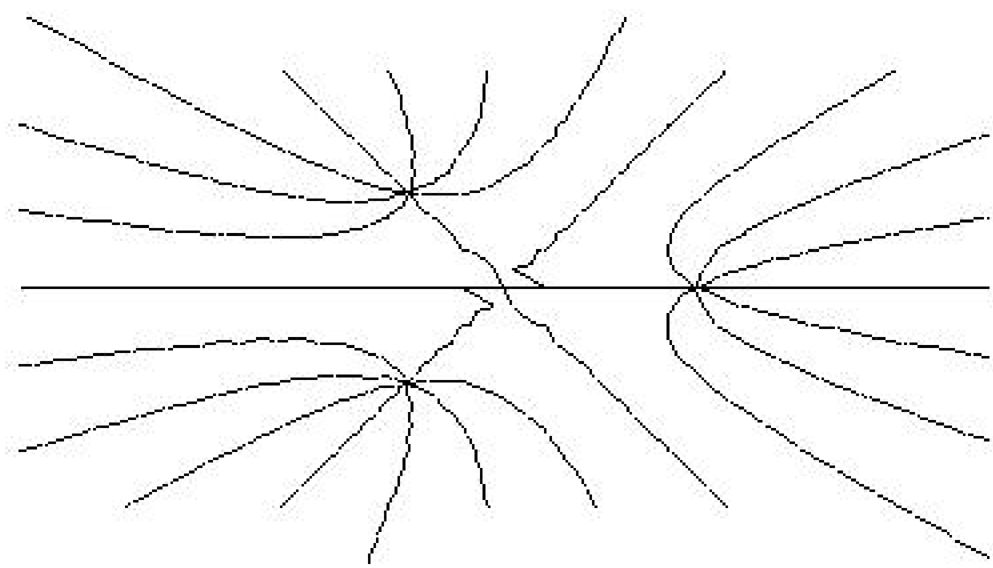
James Moor proves the following theorem, considerably stronger than the FTA, in his thesis.

Theorem 3.5.1 *Let f(z) be a non-constant complex polynomial having no multiple roots. Then the sets $\Gamma_\phi(f)$ are non-empty and, for all but finitely many $\phi, -\pi/2 < \phi \leq \pi/2$ every component of $\Gamma_\phi(f)$ contains a root of $f(z)$.*

Actually it is almost certainly true that all the components of all Γ_ϕ pass through a root, but Moor's proof fails for those values of ϕ such that there is a $z \in \Gamma_\phi$ with $f'(z) = 0$.

We will only sketch the proof here. We can show that the sets Γ_ϕ are not empty (for all ϕ, no exceptions) by approximating $f(z)$ by z^n where $\deg f(z) = n$. Now pick ϕ such that $f'(z) \neq 0$ for $z \in \Gamma_\phi$. Write $f(z) = u(z) + iv(z)$ where $u(z) = \text{Re } f(z)$ and $v(z) = \text{Im } f(z)$, then define a function $h_\phi(z) = \frac{\sqrt{2}}{2}(u(z) * \cos\phi + v(z) * \sin\phi)$. $h_\phi(z)$ is a real valued function of the complex variable z. We note that $|h_\phi(z)| = |f(z)|$ on Γ_ϕ so it is enough to show that $h_\phi(z)$ is zero for some z in each component of Γ_ϕ. (Note the similarity with Gauss's fourth proof: Gauss showed that each component of the seashore $\Gamma_{\pi/2}$ contained a point where $h_{\pi/2}(z) = \text{Im } f(z) = 0$.) The hard part of the proof is showing by means of topology and analysis that on any component of $\Gamma_\phi, h_\phi(z)$ can be neither bounded above nor below and thus takes both positive and negative values. The proof concludes by applying the intermediate value theorem which holds for any continuous real valued function defined on a connected set.

Figure 3.4: $\Gamma_0, \Gamma_{\pm\pi/4}, \Gamma_{\pi/2}$ for $z^3 - 1$



There are infinitely many Γ_ϕ, (even forgetting the exceptional ones) and each has many components, but there are only finitely many roots. It follows that many different components intersect at the roots. In fact, it follows easily from the definition of Γ_ϕ that different Γ_ϕ can only intersect at roots. Conversely, it is not difficult to prove that each root is contained in one component of Γ_ϕ for every ϕ. Thus one can locate the roots as the places where components of different Γ_ϕ intersect. This was exactly Gauss's approach to his first proof.

The picture in the figure shows the curves Γ_ϕ for $f(z) = z^3 - 1$. We will call these Gauss curves. Note that there are three places where two curves seem to intersect. These points are actually the critical points, i.e. the points where $f'(z) = 0$, and both curves are actually algebraic branches of the same Γ_ϕ. The other 4 points where many curves intersect are the roots. Although this method for finding roots is not recommended, this picture shows that with a computer Gauss's idea for proving existence of roots can actually be used to find roots.

Maple Implementation

You may wish to plot some Γ_ϕ yourself. Define a complex polynomial, eg.

```
f:=z^3 -1;
```

and then define real valued functions $u(x, y), v(x, y)$ giving the real and complex parts of $f(x + iy)$ by (using arrow notation)

```
u := (x,y) -> evalf(evalc(Re(subs(z=x+I*y,f))));
v := (x,y) -> evalf(evalc(Im(subs(z=x+I*y,f))));
```

Give the command `with(plots,contourplot);` to obtain the `contourplot` procedure from the plotting package. Pick an angle ϕ and define $r := \sin\phi, s := \cos\phi$. To plot the curve Γ_ϕ use the command

```
contourplot(r*u(x,y)-s*v(x,y), x=a..b, y=c..d,
       contours=[0]);
```

where $\{z = x + i * y | a \le x \le b, c \le y \le d\}$ is the region of the complex plane you want to see. If you want to draw several different Γ_ϕ so as to obtain a picture similar to the cover picture you should draw each separate Γ_ϕ separately and save to a variable, eg.

```
P1:=contourplot(  [stuff]  ):
```

Note the importance of the colon rather than semicolon at the end, the colon supresses printing and you do not want to see pages of raw plotting data on your screen! You also should retrieve the `display` procedure using the command `with(plots, display);`. When you have successfully saved the individual Γ_ϕ you can now draw the entire picture using the command

```
display({P1,P2,P3,P4});
```

Exercise 3.5.1 Draw the Gauss curves Γ_ϕ for the polynomial $f(z) = z^4 + 4$ for $\phi = 0, \pm\pi/4, \pm\pi/2$. Can you see the roots?

3.6 Connection between the FTA and Newton's Method

Newton was concerned with finding a method to calculate approximate roots of real polynomials whereas Gauss was interested in proving only the existence of complex roots. It comes therefore as another example of the remarkable internal consistency of mathematics that Newton's method and Gauss's proof find roots the same way – they travel along the curves Γ_ϕ.

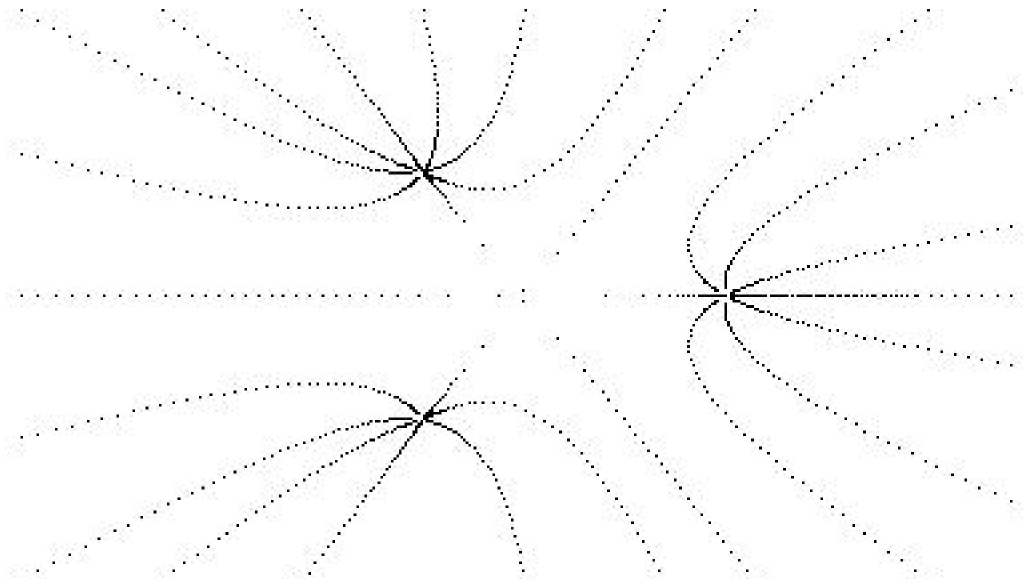
For Newton's method this is not precisely true but by experimentation one can observe a definite pattern as to how the complex Newton's method approaches a root. Starting with an initial point z_0 and defining $z_{n+1} = z_n - f(z_n)/f'(z_n)$ as in Newton's method, it can be seen that very often the values $f(z_0), f(z_1), \ldots$ approach 0 in a straight line, i.e these numbers all have approximately the same argument, say ϕ. Then the points $z_0, z_1, z_2, \ldots$ all lie in (or near) Γ_ϕ

Of course this is a tendency, not a sure thing, and hence we can not "prove" it. We can give a heuristic argument. If c is a root of $f(z)$ the Taylor's series about c is $f(z) = b_1(z-c)+b_2(z-c)^2+\cdots+b_n(z-c)^n$. When z is far away from c the top degree term dominates, when z is close to c the lowest degree term dominates. In both of these cases $f(z)$ can be approximated by a monomial $b_k(z-c)^k$. Applying Newton's method to the monomial $f(z) = (z-c)^k$ we can easily calculate $z_{n+1} = (1-1/k)z_n + c/k$ and then $f(z_{n+1}) = (1-1/k)^k f(z_n)$. Thus in this case $f(z_n)$ and $f(z_{n+1})$ have the same argument.

We can better observe this phenomenon by replacing Newton's method by a "damped" Newton's method. Here we choose a real number $h > 0$, usually h is close to 0, and starting from an initial estimate z_0 we define z_{n+1} by

$$z_{n+1} = z_n - h * f(z_n)/f'(z_n).$$

Figure 3.5: Damped Newton's method, $f(z) = z^3 - 1, h = .1$



When h is small, this slows down convergence of Newton's method and with an appropriate graphics program one can use this damped Newton's method to draw very close approximations to the curves Γ_ϕ.

In some cases when a polynomial has many roots in a small area, the damped method may actually outperform the regular Newton's method.

Maple Implementation

The figure in this section was drawn by MAPLE using the following:

```
f:=z^3-1;
f1:=3*z^2;
h:=0.1;
NC:=proc(u)
  local L,w,n;
  L:=[];
  w:=u;
  for n to 50 do
    L:=[op(L), [evalc(Re(w)), evalc(Im(w))] ];
```

```
      w:=w-h*evalf(subs(z=w,f))/evalf(subs(z=w,f1));
    od;
    RETURN(L);
  end;

  plot({NC(2.5),NC(2.5+.5*I),NC(2.5+1.5*I),NC(2+2*I),
    NC(1.15+2*I), NC(.7+2*I),NC(2*I),NC(-.5*2*I),
    NC(-1.1+2*I),NC(-1.8+2*I), NC(-2.5+2*I),NC(-2.5+I),
    NC(-2.5),NC(-2.5-I),NC(-2.5-2*I), NC(-1.8-2*I),
    NC(-1.1-2*I),NC(-2*I),NC(.7-2*I),NC(1.15-2*I),
    NC(2-2*I),NC(2.5-1.5*I),NC(2.5-.5*I),NC(-0.2+.35*I),
    NC(-0.2-.35*I), NC(0.4)},x=-2.5..2.5, y=-2..2,
    style=POINT,symbol=POINT,color=black,axes=NONE);
```

Here f is the polynomial, $f1$ is the derivative, h is the damping factor, and NC is a procedure which takes initial point u and produces a list of the first 50 iterations of the complex Newton's method.

Exercise 3.6.1 Draw the curves of the damped Newton's method for $f(z) = z^4 + 4$.

If one lets h go to zero in the damped Newton's method we get the *continuous* Newton's method which gives us precisely the Gauss curves. The idea is that instead of indexing our iterates by positive integers and writing $z_{n+1} = z_n - h\frac{f(z_n)}{f'(z_n)}$ we index by real numbers, so our iterates are $z(t)$ for $t > 0$ and $z(t+h) = z(t) - h\frac{f(z(t))}{f'(z(t))}$ This gives

$$z(t+h) - z(t) = -h\frac{f(z(t))}{f'(z(t))}$$

and then

$$\frac{z(t+h) - z(t)}{h} = -\frac{f(z(t))}{f'(z(t))}$$

and taking the limit as $h \to 0$

$$z'(t) = -\frac{f(z(t))}{f'(z(t))}$$

so

$$f'(z(t))z'(t) = -f(z(t))$$

which by the chain rule gives

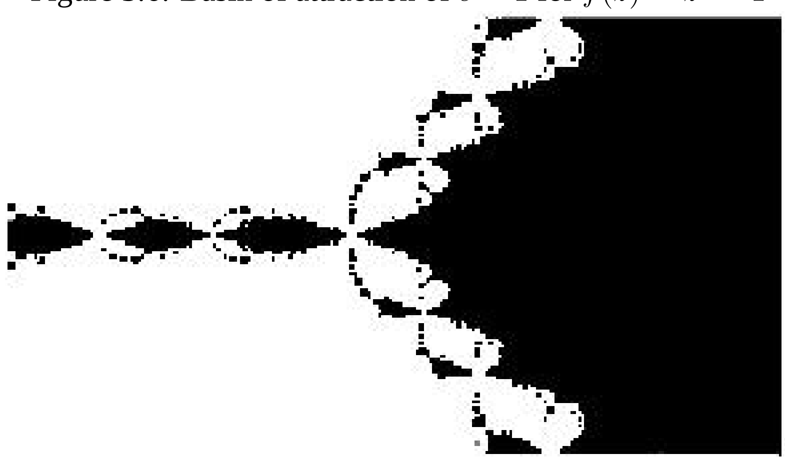
$$f(z)'(t) = -f(z(t))$$

The last equation is a differential equation which we solve with initial condition $f(z(0)) = z_0$. J.W. Neuberger shows in his article *Continuous Newton's Method for Polynomials* that the solution curves are exactly the portion of the curve $\Gamma_{\arg p(z_0)}$ from z_0 to the appropriate root. If you are experienced in solving differential equations numerically with Maple you can easily draw these curves (see Figure 3.6), although you need to view this complex equation as a system of two real equations.

3.7 Where Newton's Method does not Converge

It would seem from the discussion in the last section that given appropriate knowledge of the curves Γ_ϕ one could accurately predict which initial point will converge to which root by Newton's method. Given a root c of $f(z)$, we will call the set of complex numbers z_0 for which Newton's method with intial point z_0 converges to c, the *basin of attraction* for c. Thus portions of the curves Γ_ϕ, where ϕ is an argument of $f(c)$ for points c where $f'(c) = 0$, appear to form the boundary of the basins of attraction. This is to some extent true using a heavily damped version of Newton's method, which is why the damped method may be useful in practice. But using an undamped version of the method the boundaries of the regions of convergence usually turn out not to be algebraic curves at all, or even curves for that matter, but rather fractals.

Figure 3.6: Basin of attraction of $c = 1$ for $f(z) = z^3 - 1$



It is not difficult to show that for a given root c that its basin of attraction is an open set, further for different roots c these basins clearly have no points in common. It is more difficult to show (using advanced ideas from Topology) that the complement of any union of two or more open sets with no points in common must be an uncountable set. Since the complement of the basins of attaction is the set of points which converge to no roots this says that we have an uncountable number of such points, i.e. as many such points as there are real numbers.

This set of non-convergence, which we call the *Julia set*, is often a *fractal*. This is a set which is too big (in this case) to be one dimensional but not large enough to be two dimensional.

Fractals were discovered in the early part of this century by the mathematicians Fatou and Julia but were considered too abstract to be of interest to any but the most abstract mathematicians. In the last 20 years fractals have been rediscovered and are being studied by applied mathematicians, physicists, chemists, biologists and, mostly, by computer hobbyists. If you are interested in learning more about fractals, type the word "fractal" in any web search engine and you will get a long list of sites to visit. The Julia set of Complex Newton's method is an area of intense current study in mathematics.

With all these many points for which Newton's method does not converge one might think that Newton's complex method would be a bad method in practice. But each point at where the method does not converge is generally surrounded by points at which the method does converge. On a computer complex numbers are approximated by floating point numbers, thus even if one tries to pick a point in the Julia set as an initial point, the floating point approximation is probably not in the Julia set due to the small roundoff error. And even if it is, the next iteration probably won't be. Thus in practice

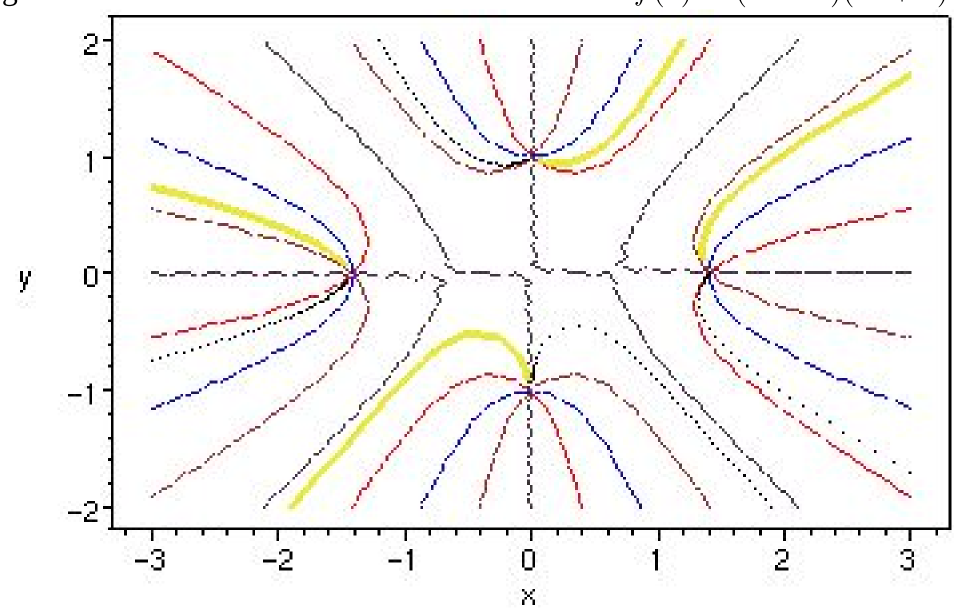
you would be quite unlucky to choose an imaginary initial point where Newton's method does not converge.

However, it is possible to give examples of polynomials and imaginary initial points where Newton's method does not converge. Generally these points are points where the Newton iteration function $g(z) = z - f(z)/f'(z)$ has an attracting periodic point. A standard example is the polynomial $f(z) = z^3 - z + .70711$ with initial point $-1.75 + 2.6i$. Another example comes from an exercise in a standard Calculus text, $f(z) = z^4 + 2z^3 - z - 1$ for which $0, -1$ are attracting periodic points of period 2 for $g(z)$. Here if you pick an imaginary initial point near 0 or -1, Newton's method won't converge.

Exercise 3.7.1 Consider $f(z) = 3z^4 - 6z^3 - z^2 + 4z - 4$. Find imaginary initial points where Newton's method does not converge. Try to describe the set of non-convergence as well as you can.

We summarize what we know about how Newton's method should and does work. In theory, the Gauss curves Γ_ϕ of section 3.5 describe the behavior of Newton's method. In particular the *critical* Gauss curves for $\phi = \arctan(v_j/u_j)$ where $z_1, \ldots, z_{n-1}$ are the critical points $f'(z_j) = 0$ and $u_j + iv_j = f(z_j)$ contain the theoretical boundaries for the basins of attraction of the roots of $f(z)$ (We are assuming no multiple root). This is precisely true when we replace Newton's method with the continuous Newton's method and almost true when we use a heavily damped Newton's method. See Figure 3.7 where the dashed curves are Gauss curves, the solid curves are from continuous Newton's Method and the dotted curves are heavily damped Newton's method.

Figure 3.7: Gauss curves and Newton's method for $f(z) = (z^2 - 2)(z^2 + 1)$



For the standard Newton's method things do not work quite as nicely. The critical Gauss curves identified above do still tend to delineate the basins, however they are surrounded by small fractal bounded regions where Newton's method behaves unpredictably and at times chaotically. As long as one stays away from these regions Newton's method will work well. Of course, it is much harder to draw the critical Gauss curves than to apply Newton's method so in practice we just apply Newton's method with more or less randomly chosen imaginary initial points and hope for the best. We are rarely disappointed.

3.8 Real Newton's Method Revisited

For real Newton's method there is a fair amount known about the Julia set and the strange behavior possible from the real Newton's method. We mention, without proof, three theorems. The interested reader may wish to look at the excellent article by Donald Saari and John Urenko in the *American Mathematical Monthly*, Volume 91, Number 1, January 1984 pp 3 - 18.

For the first theorem we need some more definitions from topology. A point x_0 is an *accumulation point* of the set D if for each $r > 0$ there is a point y of D, $y \neq x_0$ such that $|y - x_0| < r$. The difference between a limit point and an accumulation point is that a point in D is automatically a limit point of D but not necessarily an accumulation point. A set D is *completely disconnected* if each component consists of exactly one point. In the case of a subset D of the reals, D is completely disconnected if given $x < y$ both in D there exists a point z not in D such that $x < z < y$. Finally a set D is called a *Cantor* set if it is closed, every point of D is an accumulation point of D and D is completely disconnected. It can be shown that every Cantor set is uncountable, i.e. cannot be put in a 1-1 correspondence with the integers. On the other hand, Cantor sets have measure (length) zero.

The standard example of a Cantor set is the set known as THE Cantor set. Start with the unit interval $\{x|0 \leq x \leq 1\}$ and remove the open interval consisiting of the middle third, i.e. remove $\{x|1/3 < x < 2/3\}$. Then remove the middle third of each of the remaining intervals and then the middle third of the remaining intervals and so on and so on$\ldots$ What is left is THE Cantor set.

Some calculus textbooks give the impression that the only points where Newton's method does not converge is where the derivative is 0. The first theorem (due to B. Barna) says that these few points are insignificant compared to the full set of non-convergence.

Theorem 3.8.1 *Let $f(x)$ be a real polynomial of degree $n \geq 4$ with with n distinct real roots. Let D be the set of initial points for which Newton's method neither converges nor hits a point where $f'(x) = 0$. D is a Cantor set.*

Suppose $x_0, x_1, x_2, \ldots$ is a sequence of points. The sequence is called *periodic* of period k if $x_{j+k} = x_j$ for every $j = 0, 1, 2, \ldots$. In this case the set $\{x_0, x_1, \ldots, x_{k-1}\}$ is called the *orbit*. An example is the polynomial $f(x) = 1 + 4x - 12x^2 + 13x^3 - 6x^4 + x^5$ of section 2.10. Newton's iteration starting with $x_0 = 5.048177$ will give rise to a sequence which is periodic of period 8. Our next theorem which has been attributed separately to both Sarkovskii and Yorke says such examples are common:

Theorem 3.8.2 *Let $f(x)$ be a real polynomial with at least 3 distinct real roots. Then for every $k > 1$ there is an initial point x_0 so that the sequence given by Newton's method with initial point x_0 has period k.*

In §2.9 we noted that an iteration process, such as Newton's method, can have 4 outcomes, 1) it can converge to a fixed point (root in this case) and this happens for most initial points using Newton's method, 2) it can go off to infinity, which is impossible with Newton's method (although one can count those initial points which lead to an iteration where $f'(x_n) = 0$ as being in this category), 3) it can lead to a periodic obit, here the last theorem says we do get orbits of every size, and finally 4) it can give chaos. The last theorem due to Saari and Urenko says that we have any "chaos" we want. We recall from section 2.5 that if a polynomial $f(x)$ of degree n has n distinct real roots then $f'(x)$ has $n - 1$ distinct real roots, one between each pair of roots of $f(x)$.

Theorem 3.8.3 *Let $f(x)$ be a real polynomial of degree n with n distinct real roots, $n \geq 4$. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1}$ be the roots of $f'(x)$. Denote by $I(j)$ the interval $\{x | \alpha_j < x < \alpha_{j+1}\}$ for $j = 1, 2, \ldots, n-2$. Let $\{s_j\}$ be any infinite sequence of integers from the set $\{1, 2, \ldots, n - 2\}$. Then there is an initial point x_0 such that the sequence of Newton iterates satisfies $x_j \in I(s_j)$ for every $j = 0, 1, 2, 3, \ldots$.*

Once again we emphasize that these three theorems give the "theoretical" structure of the Julia set. In practice it is usually impossible to actually achieve these points by computer.

Exercise 3.8.1 The polynomial $f(x) = x^4 - 4x^3 + 4x^2 - 0.5$ has 4 distinct real roots and $f'(x)$ has roots $0, 1$ and 2. Consider the sequence

$$\{s_j\} = \{1, 2, 1, 1, 2, 1, 2, 1, 1, 1, 2, 1, 2, 1, 1, 1, 1, 2, \ldots\}.$$

Illustrate Theorem 3.8.3 by finding an initial point x_0 so that the sequence of Newton iterates satisfies $x_j \in I(s_j)$ for $j = 0, 1, 2$, if you are more ambitious do , $j = 0, 1, 2, 3, 4$, better yet $j = 0, 1, 2, \ldots, 6$. You may be able to go further. Note that in the second case you must find x_0 such that $0 < x_0, x_2, x_3 < 1$ and $1 < x_1, x_4 < 2$. It is suggested that you approach this problem both graphically and numerically and use 20 or more digit accuracy on MAPLE. [Note: Student William Browder found an example in 1999 with $x_j \in I(s_j)$ for $j = 1, \ldots, 12$ and Timothy Basaldua found one in 2001 that satisfies the condition for $j = 1, \ldots, 18$. Mr. Basaldua's example was given to 52 significant digits!]

Although one could infer from Theorem 3.8.1 that the chaotic set is uncountable we get a fairly direct proof from Theorem 3.8.3. In the setting of that theorem shift the indexing down by 1, that is say $J_0 = I(1), J_1 = I(2), \ldots, J_{n-3} = I(n-2)$. Given any sequence $s_1, s_1, \ldots$ from the set $\{0, 1, 2, \ldots, n-2\}$ we can view this as the digits in the base $b = n - 2$ expansion of the real number $s = \sum_{j=0}^{\infty} s_j b^{-j}$. Note $0 \leq s < 1$ and, conversely, every number in that interval has such an expansion. For each initial point x_0 whose iteration sequence stays in the big interval $\alpha_1 < x_j < \alpha_{n-1}$ we can then assign a real number s so that the jth digit of the base b expansion of s is k if and only if $x_j \in J_k$. The Saari-Urenko theorem then says that this function is onto. But, in particular, the behavior of Newton's method with intitial point x_0 is chaotic precisely if s is irrational! Thus there are at least as many initial points giving chaotic behavior as there are irrational numbers between 0 and 1. Thus there are uncountably many initial points giving chaotic behavior.

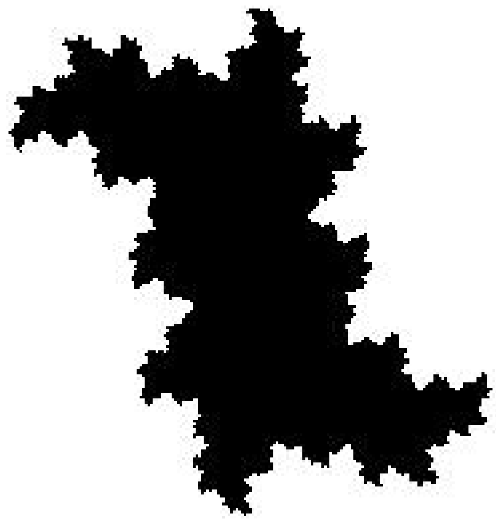
3.9 Iteration of Quadratic Polynomials

The iteration function associated to Newton's method is $g(x) = x - f(x)/f'(x)$ which is a rational, not a polynomial function. However iterating polynomial functions can give the same type of strange behavior that we observed in the previous section. In fact we don't have to go farther than the quadratic function $f(z) = z^2 + c$ to get interesting examples.

As mentioned in section 2.9, when a function is iterated the iterates may converge to a fixed point, the iterates may eventually become periodic, the iterates may go towards infinity in modulus or we may have chaos. For this function $f(z) = z^2 + c$ if we hold the parameter c fixed and vary the initial point then the set of initial points that give non-constant period behavior or chaos is properly called the *Julia set*. This set is generally a fractal. Recall a *fractal* is a set that is too big to be one dimensional but too small to be two dimensional. Unfortunately, it is difficult to draw a fractal on the computer, thus people generally draw instead the set of initial points for which the iterates do not go to infinity. This set, by a slight abuse of notation, is usually also called the Julia set, although in fact it is really the boundary of this set that is the Julia set. A typical Julia set for a polynomial of the form $f(z) = z^2 + c$ is given in the first figure.

On the other hand, we could leave the initial point constant (say at $z_0 = 0$) and vary c. Then the set of complex numbers c for which the iteration produces non-constant periodic behavior or chaos is called *the Mandelbrot set*. This is again a fractal. Again, in practice it is easier to draw instead the set of points c for which iteration starting a 0 stays bounded so it is this Mandelbrot set, one picture of which is given in the second

Figure 3.8: Julia set for $f(z) = z^2 + c$ for a typical c



figure, which has become a well known sight recently.

Again we mention that MAPLE is not a good environment for drawing Julia and Mandelbrot sets. But there is a lot of software for this available on the internet, as well as many pictures of Julia and Mandelbrot sets.

Exercise 3.9.1 Explain why the Julia set above has a rotation symmetry while the Mandelbrot set above has a reflection symmetry.

Exercise 3.9.2 Look up Julia and Mandelbrot sets on the Web and write a short report. Print out some of the pictures you find.

Figure 3.9: The Mandelbrot set for $f(z) = z^2 + c$