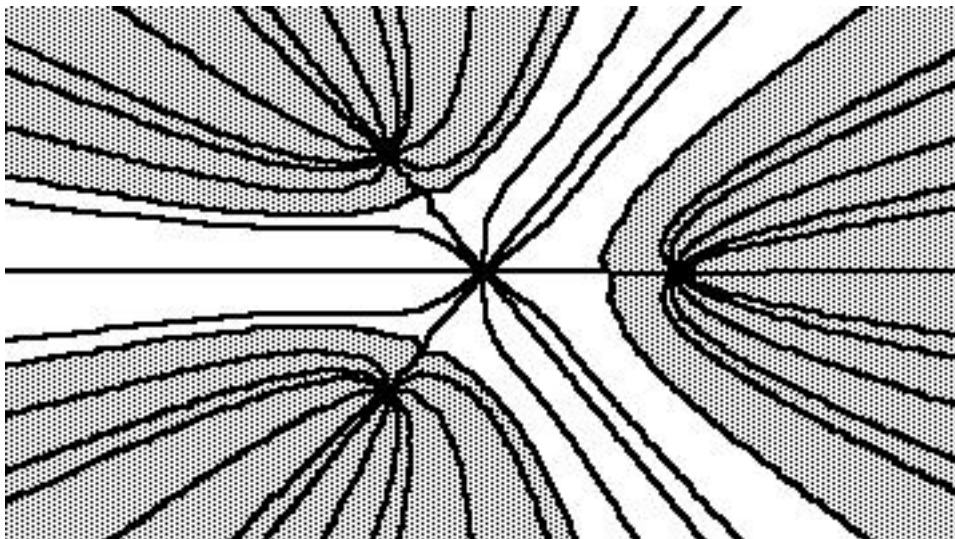# Theory of Equations

Lesson 5

by

Barry H. Dayton
Northeastern Illinois University
Chicago, IL 60625, USA

`www.neiu.edu/~bhdayton/theq/`

2.7 Graphing

A good way to find the approximate location and number of real roots is to use a computer program to graph the polynomial. In this section we will describe some of the things you should look for in the graphs. We will consider real polynomials only and we will use results from calculus.

Maple Implementation

The easiest way to plot a polynomial is to define the polynomial as an expression and then use `plot`, for example

```
f := x^4 -3*x^3 + 2*x^2 - x + 1;
plot(f, x=-1..3);       #note x-range only is specified.
plot(f, x=-1..3,y=-15..15);   # here both are specified
```

If you only specify the x range the y range is then determined automatically. Various options can be selected from the graphics toolbar (click on picture to get toolbar) such as a choice of axes style. The "1:1" option which gives `constrained` or `unconstrained` scaling should usually be left at the default setting of `unconstrained` for polynomials as different units in the x and y variables are usually necessary to plot polynomials.

Several polynomials can be plotted on the same axes, for example

```
plot({f,g,h}, x=-3..3);
```

plots the three polynomials `f,g,h` on the same axes in different colors. You may have to think a little bit, or plot the functions first individually, to figure out which is which.

Alternatively you can save each single plot under a variable name, specifying a color, and then display the plots on one graph. Do do this use the following: (output is not given)

```
with(plots,display);
P1:=plot(f, x=-3..3, color=red):
P2:=plot(g, x=-3..3, color=green):
P3:=plot(h, x=-3..3, color=blue):
display({P1,P2,P3});
```

Some comments are in order. The first command, `with(plots, display);` needs to be done only once each Maple session, the `display` procedure is in the Plots package and not automatically available for use. Note the definitions of `P1,P2,P3` are ended by colons, not semicolons. This is so the data will NOT display on your screen. For other colors see the help help page `?plot[color]` and for other available options see the help page `?plot[options]`.

We first observe that the behavior of the polynomial for large x depends only on the highest degree monomial and the behavior near the origin depends on the lowest degree non-zero monomial. We can assume without loss of generality that our polynomial is monic.

Theorem 2.7.1 *Let $p(x)$ be a monic polynomial of degree $n > 0$. Then if x is a large positive number $p(x)$ is a large positive number, if x is negative with a large absolute value, $p(x)$ is a large positive number if n is even, a large negative number if n is odd.*

Proof: This follows from the fact that

$$\lim_{x \to \infty} \frac{p(x)}{x^n} = 1.$$

Theorem 2.7.2 *Let $p(x)$ have a root of multiplicity m at $x = c$. If m is even either $p(x) \geq 0$ near x or $p(x) \leq 0$ near c. If m is odd then $p(x)$ takes on positive values on one side of c and negative values on the other.*

Proof: With the help of a Taylor's series expansion about c we have

$$\lim_{x \to c} \frac{p(x)}{(x - c)^m} = a_m.$$

Since polynomials are continuous functions the Intermediate value theorem of calculus says

Theorem 2.7.3 *Let α, β be real numbers. If $p(\alpha) < 0$ and $p(\beta) > 0$ then $p(x)$ has at least one root between α and β.*

Combining this with 2.7.2 and 2.7.1 we have the following

Theorem 2.7.4 *Let α, β be real numbers. If $p(\alpha) < 0$ and $p(\beta) > 0$ then $p(x)$ has an odd number of roots between α and β, counted according to multiplicity. In particular a polynomial of odd degree has an odd number of real roots, and a polynomial of even degree has an even number of real roots, counted according to multiplicity.*

Something to look for in the graph of a polynomial are turning points. These are local maxima or minima and correspond to values of x where $p'(x)$ has a zero of odd multiplicity. Since the degree of $p'(x)$ is always one less than that of $p(x)$ we have

Theorem 2.7.5 *A polynomial $p(x)$ of even degree has an odd number of turning points, and a polynomial of odd degree has an even number turning points.*

Rolle's Theorem gives

Theorem 2.7.6 *If α, β are real roots of $p(x)$, $\alpha < \beta$ there is at least one turning point c, $\alpha < c < \beta$.*

A zero of multiplicity greater than one is also a zero of $p'(x)$ therefore we get:

Theorem 2.7.7 *If $p(x)$ has real roots in an interval $\alpha < x < \beta$ whose multiplicities add up to m, then $p'(x)$ has at least $m - 1$ real roots c with $\alpha < c < \beta$.*

The function $p(x)$ is *concave down* for $\alpha < x < \beta$ if for every $\alpha < x_1, x_2 < \beta$ the line from $(x_1, p(x_1))$ to $(x_2, p(x_2))$ lies below the graph of the function. *Concave up* is defined similarly. In calculus we learn that $p(x)$ is concave down in an interval if $p''(x) < 0$ in that interval and concave up where $p''(x) > 0$. Points where the graph changes from concave down to concave up are called inflection points, and are points where $p''(x)$ has a root of odd multiplicity. Applying our earlier theorems of this section to $p'(x)$ we obtain

Theorem 2.7.8 *If $p(x)$ is of even degree then $p(x)$ has an even number of inflection points and is concave up for positive or negative x of large absolute value, if $p(x)$ is of odd degree ≥ 3 it has an odd number of inflection points, is concave down for negative x of large absolute value and concave up for large positive x. Finally, there are an odd number of inflection points between every pair of turning points.*

We illustrate these concepts in Figure 2.2.

Exercise 2.7.1 [10 points] Graph the following polynomial identifying roots (estimate multiplicity), turning points, and inflection points. Do not solve for roots etc. $p(x) = -2 - 2x + 5x^2 + 5x^3 - 4x^4 - 4x^5 + x^6 + x^7$.

Figure 2.2: Graph of a polynomial function
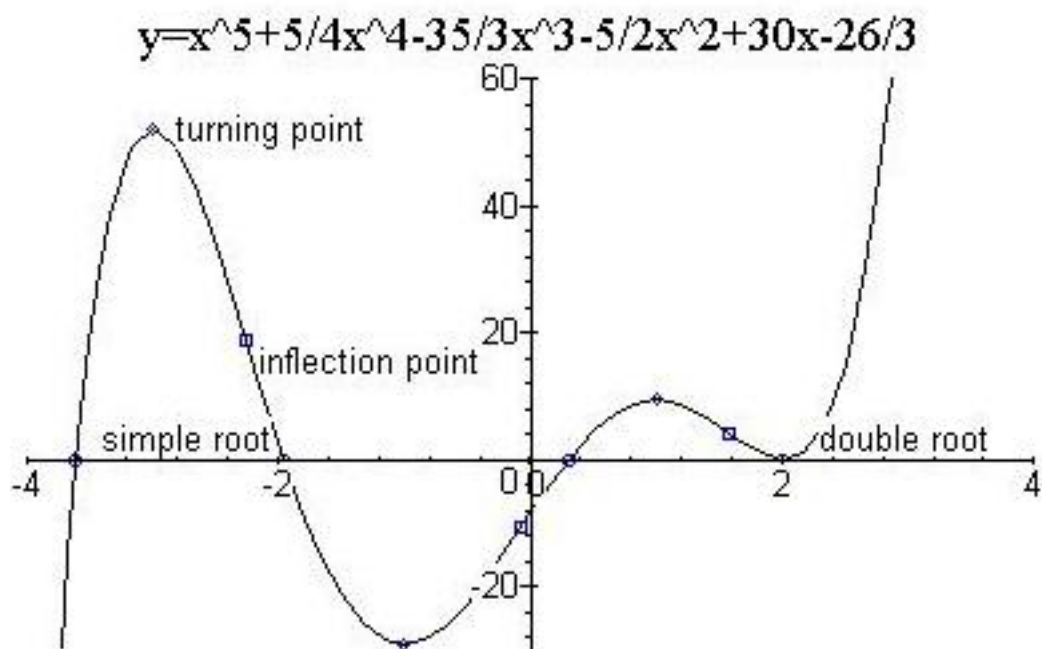
2.8 Descartes' Rule of signs

Descartes was the first to explicitly state this rule which appears in Chapter 3, the theory of equations chapter, of his famous La Geometrie. It has been suggested that the general content of this rule was known earlier by the English mathematician Harriot (the first mathematician to set foot in America) and also by Cardano. In fact, Descartes probably inferred this rule from Cardano's "Ars Magna." Cardano did not believe in negative coefficients, and so Cardano would write a polynomial equation as $x^3 + 6x = 20$ (or as Cardano put it without algebraic notation "let the cube of GH and six times the side of GH equal to 20". Thus Cardano has 13 different chapters dealing with the 13 types of cubic equations he identified and in each of these chapters he shows how to find the positive roots, and proves his method by Euclidean geometry. Descartes was the first major mathematician to write equations in our modern form as $x^3 + 6x - 20 = 0$ and so have only one type of cubic equations. He was thus the first to be able to state the rule of signs as we now do. Descartes was willing to consider negative roots but called them "false" roots, in counting roots therefore he only considered positive real roots.

In the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ we say that we have a change of signs if a_j is positive and a_{j+1} is negative or vice versa. Thus in $p(x) = -3 + 2x + 3x^2 - 4x^3 + x^4$ we have 3 changes of sign. If one coefficient is 0 we skip it, thus $p(x) = -3 + 2x^3 + 3x^5 - 4x^7 + x^11$ also has 3 changes of sign.

Theorem 2.8.1 (Descartes' Rule of Signs) *The number of positive real roots, counted according to multiplicity, of a real polynomial differs by an even number from the number of changes of sign and is never more than the number of changes of sign.*

Proof: We can assume without loss of generality that $p(x)$ is monic and $a_0 \neq 0$ (why?). Thus if $p(0) = a_0 < 0$ since the leading coefficient is 1 there must be an odd number of changes in sign, but by Theorem 2.7.4 (letting $\alpha = 0$ and β be large) $p(x)$ has an odd number of positive roots. A similar argument holds if $p(0) > 0$. Thus the number of changes in sign has the same parity (even or odd) as the number of positive roots.

We now argue by induction, if the degree is 0 there are no roots or sign changes. Now suppose Descartes' Rule holds for polynomials of degree less than p. Let $\mathrm{prts}(f)$ denote the number of positive roots of polynomial f and $\mathrm{sch}(f)$ be the number of sign changes. We have

$$\mathrm{sch}(p) \geq \mathrm{sch}(p') \geq \mathrm{prts}(p') \geq \mathrm{prts}(p) - 1$$

where the last inequality comes from Theorem 2.7.7. But $\mathrm{sch}(p)$ and $\mathrm{prts}(p)$ have the same parity hence $\mathrm{sch}(p)$ cannot equal $\mathrm{prts}(p) - 1$ so we must have $\mathrm{sch}(p) > \mathrm{prts}(p) - 1$ which, since we are dealing with integers, implies $\mathrm{sch}(p) \geq \mathrm{prts}(p)$.

We remark that applying Descartes' rule to the polynomial $p(-x)$ we can estimate the number of negative roots. Further, for any real number c, by calculating the Taylor expansion about $(x - c)$ we can estimate the number of real roots greater than c.

Thus for $p(x) = -3 + 2x^3 + 3x^5 - 4x^7 + x^{11}$ Descartes' rule says that we have either 1 or 3 positive roots. $p(-x) = -3 - 2x^3 - 3x^5 + 4x^7 - x^{11}$ so there are 0 or 2 negative roots. 0 is not a root so there are 1, 3 or 5 real roots. In fact this polynomial has only 1 real root.

2.9 The Bisection Method

The bisection method is our first method for finding roots of a real polynomial. The advantage of this method is that it is simple to implement and can be used on continuous

functions other than polynomials. Disadvantages of this method are that it only finds real roots, it is slow, it doesn't work on roots of even multiplicity and although in theory it can give high accuracy, and this accuracy can be easily measured, in practice it is hard to acheive great accuracy.

Let $p(x)$ be a real polynomial. By graphing or trial and error find α, β so that $p(\alpha) < 0$ and $p(\beta) > 0$. By the Intermediate Value Theorem (Theorem 2.7.3) $p(x)$ has a root between α and β. Let $a_1 = \alpha$, $b_1 = \beta$ and $c_1 = (a_1 + b_1)/2$ be the midpoint. Calculate $p(c_1)$. If $p(c_1) > 0$ then there must be a root between a_1 and c_1 so let $a_2 = a_1$, let $b_2 = c_1$. However if $p(c_1) < 0$ then there must be a root beween c_1 and b_1 so choose $a_2 = c_1$ and $b_2 = b_1$. The case that $p(c_1)$ is exactly zero is so remote in practice as to not be worth considering. Thus, in either case $p(a_2) < 0$ and $p(b_2) > 0$, let $c_2 = (a_2 + b_2)/2$ be the midpoint and continue as before. Each iteration of this algorithm will halve the difference $|b_k - a_k|$ so eventually these two numbers will be close together so we let $c_k + 1$ be our final answer. The error in the root is, if our calculations have been accurate so far, is no more than $|b_k - a_k|/2 = |\alpha - \beta|/2^k$.

Exercise 2.9.1 [10 points]. Use the bisection method to find a real root of $p(x) = -1 + 2x + 3x^2 - 3x^4 + x^5$ accurate to 3 significant figures.

Exercise 2.9.2 [15 points] Write a MAPLE procedure to calculate a real root of a real polynomial inside a given interval using the bisection method.

2.10   Horner's Method

Although it is now common for numerical analysts to call any method that uses Horner's process "Horner's Method" the method that classically went by that name is one that appears to have been well known in China in the 13th century and may well date back to 5th century China. The 15th century Arabic mathematician Al-Kashi either learned of this method from the Chinese, or earlier Arabs or rediscovered it. It is likely that the 13th century Italian mathematician Fibbonacci knew this method, and he may have learned it, as he learned learned much of his mathematics, from the Arabs. In any case the French Mathematician Francois Viete definitely used the method in the year 1600. As late as 1947, when J.V. Uspensky wrote his *Theory of Equations*, Horner's method was considered the method of choice for finding real roots, especially for equations with integer coefficients, as then this method could be done with only integer arithmetic and could acheive tremendous accuracy. Today, however, this method has been replaced by other methods more suited to the computer and we discuss this method primarily for its historical interest.

As done in Uspensky we discuss the method by example. For more information on the arithmetic he used and details on this method as implemented in the 1940's, consult Uspensky's book.

We will solve the polynomial equation

$$p(x) = x^4 - 2x^3 + x^2 - 3 = 0.$$

The first step, which was done classically by the separation of roots methods, is to locate the desired root between two integers. In this case we find a root between 1 and 2.

By Horner's process we then expand $p(x)$ in powers of $(x-1)$ to get $p(x) = -3 + (x-1)^2 + 2(x-1)^3 + (x-1)^4$. We now make the substitution $x = 1 + d/10$ thinking of d as the next digit in the decimal expansion, so $p(x) = -3 + (d/10)^2 + 2(d/10)^3 + (d/10)^4$. Of course we want to find d so that $p(x) = 0$, so multiplying through by 10000 (not necessary now, but done classically to avoid fractions) we must find a root of

$$p_1(d) = -30000 + 100d^2 + 20d^3 + d^4.$$

We are assured of a root between 0 and 10 so we again try integers. By calculation $p_1(5) < 0, p_1(7) < 0, p_1(9) < 0$ (but we know $p_1(10) = p(2) > 0$ so $p_1(d)$ has a solution $9 < d < 10$. We now expand $p_1(d)$ in a Taylor's series about 9 and get $p_1(d) = -759 + 9576(d-9) + 1126(d-9)^2 + 56(d-9)^3 + (d-9)^4$. We now let $d = 9 + e/10$, thinking of e as the next digit. In particular we have $x = 1.9 + e/100$. Again clearing the denominator of fractions we see we must find a root of

$$p_2(e) = -7590000 + 9576000e + 112600e^2 + 560e^3 + e^4$$

but in this case, again by trial and error we see that the root e lies between 0 and 1. Thus we set $e = 0 + f/10$, thinking of f as the next decimal digit so that $x = 1.90 + f/1000$. Making the substitution, clearing fractions we must find a root of

$$p_3(f) = -75900000000 + 9576000000f + 11260000f^2 + 5600f^3 + f^4.$$

Here we find that the root lies between 7 and 8 so we apply Horner's process at $c = 8$ and make the substitution $f = 7 + g/10$ so that we must solve for a root of

$$p_4(g) = -83143367990000 + 9734464572000g$$
$$+ 1137789400g^2 + 56280g^3 + g^4$$

This is as far as we want to go with this example, although we could, in principle, go as far as we like. Summing up

$$x = 1 + d/10$$
$$= 1 + (9 + e/10)/10 \qquad\qquad = 1.9 + e/100$$
$$= 1.9 + (0 + f/10)/100 \qquad\qquad = 1.90 + f/1000$$
$$= 1.90 + (7 + g/10)/1000 \qquad\qquad = 1.907 + g/10000$$

so to solve for x to 4 significant digits we really only need to know if g is more or less than 5. We note that the last 3 terms are much smaller (by a factor of at least 100) than the first two (remember $g < 10$) so they will not affect the root of p_4 very much. Thus it is enough to solve the linear equation

$$0 = -83143367990000 + 9734464572000g$$

whose solution is approximately $g = 8.541134171$ so we can be sure that to 4 significant digits $x = 1.908$. But in fact we could use our decimal estimate of g to estimate $x = 1.907854113$. It turns out that this is accurate to 6 significant digits, actually to 10 significant digits $x = 1.907853262$.

If one wanted more accuracy, instead of dropping the last 3 terms of $p_4(x)$, we might only drop the last term. After several iterations of the cubic we would drop another term and do several iterations with the quadratic. Finally we would drop the quadratic term and solve the resulting linear equation. In principle, this method can give any desired accuracy. For instance in Uspensky's book there is a calculation of a root by this method to 12 significant digits.

Maple Implementation

One can use the fact that MAPLE does exact integer arithmetic to any accuracy to use Horner's method. The following sequence of MAPLE commands (output is supressed) will do the calculation of the example above:

```
> p := -3 + x^2 -2*x^3 + x^4;
> taylor(",x=1,5);
> subs(x=1+d/10,");
> convert(",polynom);
> p1 := 10000*";
> taylor(p1,d=9,5);
```

```
> subs(d=9+e/10,");
> convert(",polynom);
> p2 := 10000*";
> subs(e=f/10,p2);
> p3 := 10000*";
> taylor(p3,f=8,5);
> subs(f=7+g/10,");
> convert(",polynom);
> p4 := 10000*";
> g := evalf(tcoeff(p4,g)/coeff(p4,g,1));
```

Note that the procedure `taylor` returns the `series` data type so the command `convert(",polynom);` is used to convert back to a polynomial so the simplification procedures of MAPLE can be used.

Exercise 2.10.1 [20 points, From Uspensky's book] Use Horner's method to solve $3x^3 - 7x^2 + 2x + 5 = 0$ to 6 significant digits. Show your intermediate steps or your MAPLE printout.

Exercise 2.10.2 [20 points] Generally we think of the second derivative as a measure of how "far from linear" a function is, but that only works when the first derivative is of reasonable size. For large first derivative the *curvature* of a function at x_0 can be approximated by

$$\kappa = \left| \frac{2f''(x_0)}{f'(x_0)^3} \right|$$

Calculate the curvature at $x_0 = 0$ for the polynomials $p_1(d), p_2(e), p_3(f)$ and $p_4(g)$ using this formula. What does this say about the approximate linearity of these polynomials near 0? How does this help justify the approximation at the end of the method?

2.11 Iteration Method

The method of iteration is an important method in numerical analysis. It can be used to solve polynomial as well as other equations. The methods we will be considering later mostly fall under the category of iteration methods, so in this section we take a brief look at the general method.

Let g be a real valued function of a real variable or a complex valued function of a complex variable. A number c is called a *fixed point* of g if $g(c) = c$.

We will consider the following situation. Let x_0 be a complex number, let

$$\begin{aligned} x_1 &= g(x_0) \\ x_2 &= g(x_1) \\ x_3 &= g(x_2) \\ &\vdots \qquad \vdots \end{aligned}$$

and so on, in general $x_{n+1} = g(x_n)$.

There are 4 possibilities:

1. The sequence $\{x_n\}$ converges to a fixed point of $g(x)$.

2. The moduli of the numbers $\{x_n\}$ go to infinity.

3. The sequence $\{x_n\}$ *orbits* around a finite number of points.

4. We have *chaos*.

Possibility 3) means that $x_n = x_m$ for some $n > m$ which then implies that $x_{j+n-m} = x_j$ for all $j \geq m$. The last possibility is essentially "none of the above", we will discuss the last two possibilities further in Chapter 3. What we hope will happen, of course, is that possibility 1) will occur.

For example, let $g(x) = (x^2 + 2)/2x$. Start with just about any real (or complex) number for x_0, say $x_0 = 1$. Then $x_1 = g(x_0) = .75, x_2 = g(x_1) = g(.75) = 1.70, x3 = g(x_2) = 1.4395325$ and succeeding terms in the sequence are $1.4144362, 1.4142136,$ $1.142136, 1.142136, \ldots$ We see that this sequence has converged, in fact to $\sqrt{2}$. We note that in this case it only took 5 iterations to reach the limit.

Another amusing example is $g(x) = \sqrt{x+1}$. Starting with any positive initial point the iteration sequence will converge to the golden mean $(1 + \sqrt{5})/2$.

There are numerous theorems in mathematics which give conditions under which the sequence $\{x_n\}$ will converge to a fixed point. We are interested in the following general situation.

A fixed point c of g is called an *attracting* fixed point if $g'(c) < 1$. c is called a *repelling* fixed point if $g'(c) > 1$. For example, $1/3$ is a repelling fixed point of $g(x) = 10x - 3$ (we discussed this function briefly in §2.1). Repelling fixed points repell x away from the fixed point. However if c is an attracting fixed point, and x_0 is chosen near c then $\{x_n\}$ will converge to c. This was the case in the example $g(x) = (x^2 + 2)/2x$ above.

To apply the fixed point iteration scheme to find the root of a polynomial we rewrite the equation $p(x) = 0$ to the form $x = g(x)$, and hope that the function g has an attracting fixed point. For instance if $p(x) = x^3 + 4x^2 - 10$ then $g(x) = x$ is equivalent to $p(x) = 0$ for the following functions g:

$$
\begin{aligned}
g_1(x) &= x - x^3 - 4x^2 - 10 \\
g_2(x) &= \sqrt{10/x - 4x} \\
g_3(x) &= \frac{1}{2}\sqrt{10 - x^3} \\
g_4(x) &= \sqrt{\frac{10}{4 + x}} \\
g_5(x) &= \frac{2x^3 + 4x^2 + 10}{3x^2 + 8x}
\end{aligned}
$$

Starting with $x_0 = 1.5$ the first sequence goes to infinity, the second becomes undefined (a negative number inside the square root), but the last 3 converge to the root of $p(x)$ in 30, 15 and 4 iterations respectively.

One final comment on fixed point iteration. Suppose for a given function $g(x)$ and intitial point x_0 the sequence $\{x_n\}$ converges to an attracting fixed point c. Even if in actual computation the x_n are not calculated accurately, the slightly erroneous x_n will also be attracted to c. Thus fixed point iteration is to a large extent self correcting in terms of numerical error. Thus this method is highly accurate numerically, when it works at all.

Maple Implementation

The most convenient way to do iteration in MAPLE is to define g as an `arrow` procedure, for example to implement iteration with the example $g_4(x) = \sqrt{10/(4 + x)}$ above you would give the commands

```
g := x -> sqrt(10/(4+x));
g(1.5);
g(");
g(");
   .
   .
   .
```

One can also use a `for .. od` loop as follows

```
g := x -> evalf(sqrt(10/(4+x)));
x := 1.5;
for j to 10 do x := g(x); od;
```

Note the use of `evalf`. This is important to remember in doing MAPLE iteration as MAPLE tries to use exact expressions as possible. Iterating exact expressions is very time consuming and uses up much memory and gives a very large output. To see this start with $x = 3/2$ in the first version of iteration above. Also, be very careful not to start your iteration with a unassigned variable or algebraic expression.