

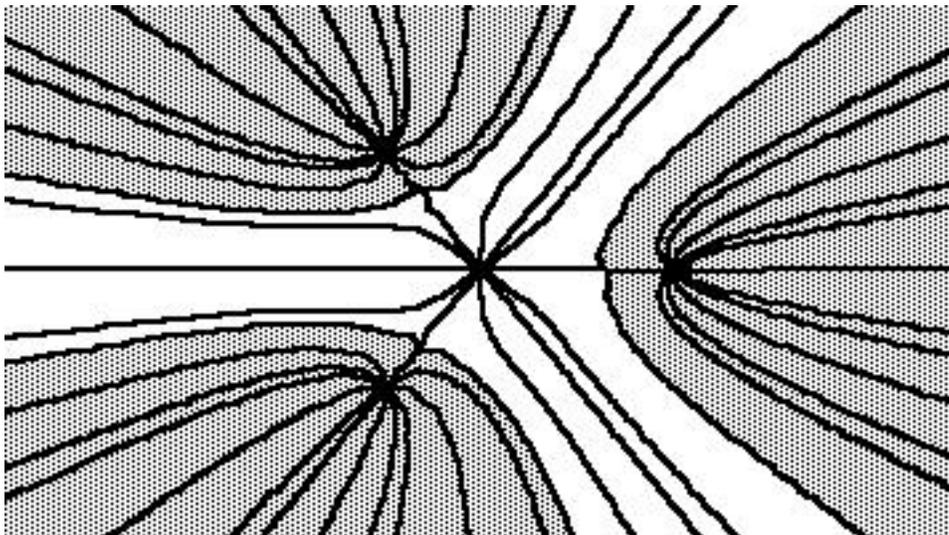
Theory of Equations

Lesson 1

by

Barry H. Dayton
Northeastern Illinois University
Chicago, IL 60625, USA

www.neiu.edu/~bhdayton/theq/



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Preface

These notes were first prepared for a course given at Northeastern Illinois University in the summer of 1989. Theory of Equations was a traditional course in the mathematics curriculum which dealt with the important topics of the theory and solution of polynomial and linear equations. Recently, thanks to the increased availability of computers, linear algebra has taken a central role in mathematics and there are specialized courses devoted to that topic. On the other hand, the theory of polynomial equations, while still important, has been scattered among various courses.

I had two motivations for preparing notes rather than using a standard text. First of all there have been many changes in the way we view the material since the 1940's when such classics such as the text by Uspensky were written. Computers, and even calculators, have made much of the traditional material, which dealt with efficient hand computation schemes and separation of roots, obsolete. Different numerical methods are now preferred. Interest in dynamical systems and chaos has brought new insights into Newton's method. And there have been exciting new algorithms developed for factoring polynomials. Thus one motivation for these notes is to give a modern view of this classical subject.

My other motivation has to do with the number of different areas of mathematics in which polynomials arise. Thus by using the concept of the polynomial as a common thread I am able to cover a large area of the mathematical landscape. Generally students of mathematics are able to see only a small portion of the total of mathematics and often are not given the motivation, practical and historical, for many of the branches of modern mathematics. Thus it is my intention to use this course partly as a survey of a large chunk of pure and applied mathematics from a historical as well as modern point of view. It is my hope that students will emerge from this course not so much with specific skills in solving polynomial equations but rather with a better perspective on what mathematics is, where it came from, and where it is going.

I wish to thank various students who have helped motivate me and make these notes possible. First of all, my Summer 1989 class for their patience in receiving these notes a few pages at a time and who pointed out many typographical and other errors. I also wish to acknowledge my debt to James Moor and Ken Touff whose masters projects provided the main motivation and much of the material for Chapter 3.

And finally I wish to give special thanks to James Moor who contributed to the preparation for this edition, especially with careful proofreading and contribution of new problems for Chapter 1.

Chapter 1

College Algebra

REAL AND COMPLEX POLYNOMIALS

In this chapter we review the standard theory of real and complex polynomials as might appear in a good high school college algebra course.

1.1 The Real Number System

We start out with an axiomatic approach to the real number system; that is, we list a number of properties that the system of real numbers possesses. In this course we will not attempt to construct this system nor justify the axioms. This is left for a course in real analysis. We will use these axioms as our starting point and base future proofs and constructions on them. We also have a second motivation: the properties of these axioms, associative laws, distributive laws, etc., will arise in many other contexts.

The *real number system* consists of a set of numbers \mathbf{R} , two binary operations $+$, $*$, and a relation $<$ which satisfy the following axioms:

1. Closure

C1. Given $a, b \in \mathbf{R}$, there exists a unique element $a + b \in \mathbf{R}$.

C2. Given $a, b \in \mathbf{R}$, there exists a unique element $a * b \in \mathbf{R}$.

2. Ring Axioms

R1. For all $a, b \in \mathbf{R}$. the *commutative law* $a + b = b + a$ holds.

- R2. For all $a, b \in \mathbf{R}$, the *commutative law* $a * b = b * a$ holds.
- R3. For all $a, b, c \in \mathbf{R}$, the *associative law* $a + (b + c) = (a + b) + c$ holds.
- R4. For all $a, b, c \in \mathbf{R}$, the *associative law* $a * (b * c) = (a * b) * c$ holds.
- R5. For all $a, b, c \in \mathbf{R}$, the *distributive law* $a * (b + c) = (a * b) + (a * c)$ holds.
- R6. There is a unique number 0 such that $a + 0 = a$ for all $a \in \mathbf{R}$. 0 is called the *additive identity*.
- R7. There is a unique number 1 such that $a * 1 = a$ for all $a \in \mathbf{R}$. 1 is called the *multiplicative identity* or *unit element*.
- R8. For each $a \in \mathbf{R}$, there is a unique $(-a) \in \mathbf{R}$ so that $a + (-a) = 0$. The number $(-a)$ is called the *additive inverse* of a .

3. Integral Domain Axiom

- I1. Given $a, b \in \mathbf{R}$, if $a * b = 0$ then $a = 0$ or $b = 0$.

4. Field Axiom

- F1. Given $a \in \mathbf{R}$ such that $a \neq 0$, there exists a unique $(a^{-1}) \in \mathbf{R}$ such that $a * (a^{-1}) = 1$. (a^{-1}) is called the *multiplicative inverse* of a .

5. Order Axioms

- O1. Given $a, b \in \mathbf{R}$, exactly one of $a < b$, $a = b$, or $b < a$ holds.
- O2. If $a, b, c \in \mathbf{R}$, then $a < b$ and $c < d$ implies $a + c < b + d$.
- O3. If $a, b \in \mathbf{R}$, then $0 < a$ and $0 < b$ implies $0 < a * b$.

6. Completeness Axiom

- O4. If S is a non-empty subset of \mathbf{R} and $M \in \mathbf{R}$ satisfies $a < M$ for all $a \in S$, then there exists $u \in \mathbf{R}$ so that $a \leq u$ for all $a \in S$, and if some $b < u$, then there is an element $a \in S$ with $b < a$. The element u is called the *least upper bound* of S .

A few comments on these axioms are in order. The “closure” axioms say that the operations of addition and multiplication are defined; that is any two real numbers can be added or multiplied. We will define the operations of subtraction and division

in terms of these basic operations. In particular, since axiom R8 says that each real number b has a unique additive inverse $-b$, we can define subtraction by

$$a - b = a + (-b).$$

In a similar manner we can define division by

$$a/b = a * (b^{-1}) \text{ when } b \neq 0.$$

Division, unlike addition, multiplication, and subtraction does not satisfy closure axioms; division by 0 is not possible. Note also that subtraction and division fail many of our laws, in particular, the commutative and associative ones.

One should also note that axiom I1 is redundant: it can be proved from axiom F1. For if $a * b = 0$ but $a \neq 0$, then multiplying by a^{-1} gives

$$b = 1 * b = (a^{-1} * a) * b = a^{-1} * (a * b) = a^{-1} * 0 = 0.$$

Axiom I1 is included here for later reference.

The relation $<$ is defined by the order axioms. One can easily define the relations $>$, \leq , \geq from this.

Finally, it is the Completeness Axiom that really distinguishes the real number system from other systems such as the rational number system. It is from the completeness property that we can define such numbers as $\sqrt{2}$, π , and e . For example, $\sqrt{2}$ can be defined as the least upper bound of the set $S = \{a \in \mathbf{R} | a^2 \leq 2\}$. The set S is bounded above by, for instance, the number 2, and S is not empty since $1 \in S$, so the least upper bound $\sqrt{2}$ exists.

1.2 A little History of Real and Complex Numbers

Although we, as in many algebra and calculus books, take the real numbers as a starting point, historically the real number system as we know it today appeared rather late in the game. In fact, only since the end of the last century have the properties of the real numbers been carefully established and axiomizations such as we have given date from early in this century.

Although the Pythagoreans started out promisingly around 500 BC with mathematics as the study of numbers, the discovery by Pythagoras, or more likely one of his followers, of the incommensurability of the diagonal of a square (that is, the fact that $\sqrt{2}$ is not a rational number) put a damper on the development of the number system.

Euclid, following the precepts of Plato and Aristotle, developed his geometry entirely without the aid of numbers, using line segments instead. Unfortunately for mathematics the Euclidean approach was dominant for over 1500 years in Europe.

The Hindu mathematicians did use numbers and it is from them that we get our current numerals. The ninth century Arab mathematician Al-Khwarizmi did advocate the use of numbers and is known as the father of algebra (*Al-jabr* was the name of his book), but his immediate followers insisted on “proving” algebra using Euclidean geometry. Fibonacci was instrumental in bringing algebra and numbers to Europe about 1200 but owing to the still powerful influence of Euclid the practice of “proving” algebraic solutions by geometry remained through the time of Cardano and beyond. It was Cardano who, in his 1545 book *Ars Magna*, first suggested the idea that an equation could have a negative solution. Descartes furthered the use of negative numbers in his famous *La Geometrie* in 1637 using negative numbers as coefficients as well as roots, but still called negative roots “false.”

With the invention of calculus late in the 17th century numbers once again became central in mathematics. Yet it was not until the middle 19th century that it was felt necessary to establish the formal properties of the real number system. These properties were finally established by the work of mathematicians such as Bolzano, Weierstrass, Cantor, and Dedekind. The present axiomatic approach is due largely to the influence of Hilbert in the early part of this century.

Complex numbers are no more recent than real numbers. In fact, complex numbers made their appearance in the very same chapter of *Ars Magna* that negative numbers first appeared. Cardano asserted that these numbers did not really exist; you had to “imagine” them. It is from this comment that non-real complex numbers are still called *imaginary*. In fact the term *real* appears to have been used first to mean “not imaginary.” This usage appears for instance in Descartes. Complex numbers were studied by Bombelli in 1572 and by Girard in 1629. Leibniz, co-inventor of calculus, was fascinated by complex numbers in the late 17th century only partly because of his belief that they could be used to give a proof of the existence of the Holy Ghost. De Moivre, a follower of Newton, studied complex numbers in 1707. Complex numbers played an important role as Euler, in the years from 1746 through 1777, established the modern terminology of mathematics with such symbols as $e, \pi, i, f(x)$ and introduced the modern trigonometric functions.

Later in this chapter and in Chapter 3 we will see the importance of complex numbers in solving equations with the Fundamental Theorem of Algebra. The further importance of complex numbers in modern mathematics was firmly established by the work of Cauchy and Riemann in the 19th century. The Fundamental Theorem of Algebra is only one of many powerful theorems in mathematics that would be lost without

the use of complex numbers. In physics, complex numbers are very important in the study of periodic behavior, such as electromagnetic waves.

1.3 Complex Numbers

We assume the existence of a number i (not in \mathbf{R}) with the property that $i^2 = -1$. The set \mathbf{C} of complex numbers consists of all numbers $a + bi$ where a, b are real numbers. In order to have laws R1 - R5 hold we are forced to define operations $+, *$ by

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ \text{and} \\ (a + bi) * (c + di) &= (ac - bd) + (ad + bc)i\end{aligned}$$

Note that $0 = 0 + 0i$ is an additive identity and $1 = 1 + 0i$ is a multiplicative identity. Furthermore $-(a + bi) = (-a) + (-b)i$ so in fact all of the rules R1 - R8 hold.

For a complex number $z = a + bi$ the real number a is called the *real part* of z , i.e., $a = \text{Re}(z)$, and the real number b is called the *imaginary part* of z . We write $b = \text{Im}(z)$. z is called *real* if $\text{Im}(z) = 0$ and *imaginary* if $\text{Im}(z) \neq 0$.

For a complex number $z = a + bi$ the complex conjugate is $\bar{z} = a - bi$. Several important properties of the conjugate should be noted:

$$\overline{z + w} = \bar{z} + \bar{w} \text{ and } \overline{z * w} = \bar{z} * \bar{w}$$

and if $z = a + bi$, then

$$z * \bar{z} = a^2 + b^2$$

is a non-negative real number.

In light of the last equation we can define the *modulus* or *absolute value* of a complex number $z = a + bi$ to be $|z| = \sqrt{a^2 + b^2} = \sqrt{z * \bar{z}}$ (as usual in mathematics, in these notes \sqrt{x} will denote the positive square root of the positive real number x). Again note the identities

$$\begin{aligned}|z * w| &= |z| * |w| \\ |-z| &= |z| \\ |\bar{z}| &= |z| \\ |z + w| &\leq |z| + |w|. \\ |z| &= 0 \text{ if and only if } z = 0.\end{aligned}$$

If r is a real number and $z = a + bi$ a complex number it makes sense to write $\frac{z}{r} = \frac{a}{r} + \frac{b}{r}i$. Thus given $z = a + bi \neq 0$ we have a complex number

$$w = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

A simple calculation shows that $zw = 1$. Thus we can write $z^{-1} = \bar{z}/|z|^2$ and we see axiom F1 holds. As in the case of \mathbf{R} we can write (for $w \neq 0$) $z/w = z * w^{-1}$. Thus the complex numbers satisfy all the field axioms that the reals do.

We note that it is impossible for us to define a relation $<$ on \mathbf{C} satisfying O1 - O4 for by O1 we must have either $0 < i$ or $i < 0$ which in the second case would give by O2 $0 < -i$. But since $i^2 = (-i)^2 = -1 < 0$, O3 would be violated. **Thus we will only use the relation $<$ when both sides are known to be real.**

Until recently doing arithmetic with complex numbers was sort of a pain since complex numbers were not supported by most calculators and most computer languages such as BASIC, Pascal etc. (FORTRAN was an exception). However many of the new generation of graphing calculators do support complex numbers, as do the symbolic mathematics programs such as Maple and Mathematica. A handy way to do complex arithmetic today is to use the TI-85/86 calculator. Although we will not use the TI-85/86 later in these notes, we will explore the complex number capabilities of these calculators since this gives insight into the arithmetic of complex numbers.

TI-85/86 Implementation

The complex constant $a + bi$ is expressed in ordered pair notation (a, b) , for instance $3 + 2i$ is entered as $(3, 2)$. To store as a variable simply use the [STO] key, i.e., $(3, 5)$ [STO] z assigns the variable z the value $3 + 5i$. Likewise you can use $(0, 1)$ [STO] i to assign the number i to the variable i . Then you can enter $3 + 4i$ by $3 + 4 * i$, but it will still display as $(3, 4)$. Note that complex variables are a special variable type on the VARS or MEM screens. Complex constants, expressions or variables can be used as list, vector and matrix elements.

I find it convenient to set the number of decimal places to 3 or 4 on the MODE screen so I can see the whole output on the screen at one time. To do arithmetic just type it in as if you were calculating with real numbers i.e.,

$$(1, 3) + (5, 7)$$

```

(2,1) * (4,-7)
(3,6) / (8,2)
z + w/u
3*(2,5)+8

```

and so forth. Note that in the last example, when mixing real and complex numbers the real numbers are evaluated as complex numbers i.e., $3 = 3 + 0i$ etc. Your TI-85/86 will even interpret the expression (X, Y) correctly if X and Y are real variables or expressions. (Note: for those unfamiliar with the TI-85/86, negative numbers are entered with the $[-]$ key, not the usual minus key.)

The `CPLX` menu has functions which calculate the conjugate, find the real and imaginary parts and the absolute value or modulus of a complex number or variable, for example you can enter expressions such as

```

conj (3,5)
real z
abs (z*w)

```

It is not an accident that the TI-85/86 uses the notation (x, y) for a complex number. Geometrically we can identify the set C with the ordinary Euclidean plane by associating the complex number $a + bi$ with the point or vector (a, b) . One can then translate some of our algebra to geometry as follows. The x -axis can now be called the *real* axis while the y -axis is the *imaginary* axis. Addition of complex numbers corresponds to vector addition in the plane. The modulus of a complex number is the length of the corresponding vector. The negative of a complex number corresponds to the negative of the vector and the conjugate of $z = a + bi$ is the reflection of (a, b) in the x -axis. The geometrical interpretation of multiplication is more complicated but even more beautiful and important than the above.

Before discussing multiplication we must discuss the *argument* or *angle* of a complex number. Given $z = a + bi$ we measure the angle from the positive real axis to the point (a, b) in radians (ALWAYS radians). This angle is allowed to be positive or negative and if one wants one may go all the way around several times. Thus it is permissible to measure the argument of the complex number i as $\pi/2, -3\pi/2, 5\pi/2$ or even $29\pi/2$. Unfortunately there is no one correct way to measure the argument, we do note however that different permissible values for the argument of a given complex number differ by $2n\pi$ where n is an integer.

Then if a complex number $z = a + bi$ has modulus $r = |z|$ and some argument α then the *polar* form of z is $z = r(\cos \alpha + i \sin \alpha)$. From the equations

$$\begin{aligned}\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta\end{aligned}$$

we see that if $z = r(\cos \alpha + i \sin \alpha)$ and $w = s(\cos \beta + i \sin \beta)$ then $z * w = rs(\cos(\alpha + \beta) + i \sin(\alpha + \beta))$. Thus multiplying complex numbers corresponds geometrically to multiplying the moduli and adding the arguments. Note also that for z, w as above

$$\begin{aligned}\frac{z}{w} &= \frac{r}{s}(\cos(\alpha - \beta) + i \sin(\alpha - \beta)) \\ -z &= r(\cos(\alpha + \pi) + i \sin(\alpha + \pi)) \\ \bar{z} &= r(\cos(-\alpha) + i \sin(-\alpha)).\end{aligned}$$

From the above geometry of multiplication we get De Moivre's famous rule (given in 1707) for finding an integral power of a complex number $z = r(\cos \alpha + i \sin \alpha)$, mainly

$$z^n = r^n(\cos n\alpha + i \sin n\alpha).$$

The importance of this rule is that one can go backwards, i.e., one can find the n th root of any complex number. However we must note carefully that because of the uncertainty in the argument there are several, in fact n , angles which when multiplied by n give equivalent arguments. Thus for $z \neq 0$, and $n > 0$ an integer the n th roots of $z = r(\cos \alpha + i \sin \alpha)$, ($r > 0$) are

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\alpha + 2\pi k}{n}\right) + i \sin\left(\frac{\alpha + 2\pi k}{n}\right)\right)$$

for $k = 0, 1, 2, \dots, n - 1$.

Of particular importance are the n th roots of unity, that is the n complex n th roots of 1. These are the numbers

$$\zeta^k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$$

These numbers are useful, for if we can find one n th root w of a complex number z , then the other roots are $\zeta^1 w, \zeta^2 w, \dots, \zeta^{n-1} w$.

TI-85/86 Implementation

The TI-85/86 has the special notation $(r\angle\alpha)$ for the complex number $z = r(\cos \alpha + i \sin \alpha)$. Note that the \angle key is [2nd][,] so \angle takes the place of the comma in the ordered pair. For example, entering $(3\angle 3 * \pi/2)$ into the TI-85/86 gives the result $(0, -3)$ for the complex number $-3i$, whereas entering $(3\angle \pi/4)$ gives the result $(2.1213, 2.1213)$. However changing the MODE screen from RectC to PolarC changes the output of each complex number to polar form. One can also use the functions Rec and Pol in the CPLX menu to change from polar to rectangular and back. Note that the syntax to change rectangular to polar form is (x, y) [CPLX][MORE][F1] .

To find an integral power of a complex number simply use the power key, [^] For example $(3, 4)^7$ gives the result $(76443, 16124)$. With the PolarC mode on, $(2\angle .7)$ raised to the 5th power gives the result $(32\angle -2.7832)$. Note that by DeMoivre's rule the argument should be $.7 * 5 = 3.5$ but the TI-85/86 prefers to give an argument between $-\pi$ and π so gives $3.5 - 2\pi = -2.7832$.

In PolarC mode it appears that taking the n th root of $(r\angle\alpha)$ gives the one value $(r^{1/n}\angle\alpha/n)$ with the angle normalized to the standard range $-\pi$ to π . Thus to get all the n th roots you need to add the angles $2k\pi/n$ to the argument for $k = 0, 1, \dots, n - 1$. The easiest way to do this is to multiply the original result by the n th roots of unity $(1\angle 2k\pi/n)$ for the appropriate values of k .

This works also in RectC mode. Thus to find the 3 cube roots of $2 + 5i$, for example, enter $(2, 5)^{(1/3)}$ (or use the special n th root key on the [MATH][MISC] menu) getting $(1.6166, .6773)$ which is one cube root. Now multiply this by $(1\angle 2 * \pi/3)$ (remember to use the internal value of π which is [2nd][^]) to get $(-1.3949, 1.0614)$. Now multiply this last value again by $(1\angle 2 * \pi/3)$ to get the third root $(-.2217, -1.7387)$. An alternate approach is to use STO and the CPLX menu to first do [CPLX][abs] $(2, 5)$ [STO] R to set R to the value $|2 + 5i| = 5.3852$ and to then do [CPLX][angle] $(2, 5)$ [STO] A to set A to the value 1.1903 which is the argument of $2 + 5i$. Then do $R^{(1/3)}$ [STO] S and the first cube root is $(S\angle A/3)$ which is $(1.6166, .6773)$ as above, the second is $(S\angle A/3 + 2 * \pi/3)$ which gives $(-1.3949, 1.0614)$ as above and the third is then $(S\angle A/3 + 4 * \pi/3)$, again the same as above.

Finally note that the n th root functions on the TI-85/86 are not

continuous. For example $\sqrt{(2\angle 3.132)}$ is $(.0068, 1.4142)$ whereas $\sqrt{(2\angle 3.153)}$ evaluates to $(.0081, -1.4142)$. This is not a defect of the TI-85/86 calculator; it can be proven mathematically that there does not exist a continuous function on C which gives one of the square roots of each number z . The TI-85/86 is a marvelous little device, but it occasionally can't do the mathematically impossible!

Exercise 1.3.1 Find (preferably using a TI-85/86) the following in the form $a + bi$, using decimal approximations when appropriate.

1. $(3 + 4i) * (4 - 3i)$
2. $(2 - 3i)/(5 + 6i)$
3. $(3 - 7i)^4$
4. The 5 fifth roots of $3 + 4i$
5. The three complex cube roots of -2
6. The four complex fourth roots of 5 .

In the future in these notes we will be using MAPLE as our calculating and computing environment. We take our first look at MAPLE now.

Maple Implementation

The syntax for MAPLE is not that different from the TI-85 with some important exceptions. Variables which have not previously been assigned numerical values are treated as *indeterminants*. Algebraic expressions in these variables are then manipulated algebraically. Also MAPLE treats rational numbers and floating point numbers differently. The quotient of two rational numbers is a fraction. Roots of rational numbers are not evaluated as floating point numbers but left in surd form. Numbers with decimal points are treated as floating point numbers, others are not.

The most important thing to remember about MAPLE are that you must end your input with a semicolon ";" unless you do not want to see output in which case you use a colon ":". Also when you assign a value to a variable use the assignment operator `:=` as in

```
x := sqrt(2);
```

to assign the variable `x` the value $\sqrt{2}$. If you want a decimal value but MAPLE returns an exact (symbolic) value use `evalf` as in

```
evalf(sqrt(2));
```

to get a decimal approximation of the square root of 2.

We now turn to complex numbers. The letter `I` (capital I) is reserved for the MAPLE $\sqrt{-1}$. Thus a complex number such as $3 + 4i$ would be written `3 + 4*I` using MAPLE syntax.

To extract the real or imaginary part of a complex number z use

```
x:=Re(z);
y:=Im(z);
```

note that to retain consistency with earlier versions of MAPLE `Re`, `Im` violate the usual case rules of MAPLE which usually start built in MAPLE procedures with small letters. The procedure `evalf` works with complex as well as real numbers to convert to floating point. A procedure `evalc` is useful for breaking an imaginary number into its real and imaginary parts. The modulus can be calculated by `r:=abs(z)`; and the argument is given by `t:=argument(z)`;

There is a provision for conversion between polar and rectangular coordinates. `polar(r,alpha)` is a name given to the complex number $z = r(\cos \alpha + i \sin \alpha)$ where `alpha` is the variable name of the numerical value of α . This representation is only known to `evalc`. Thus `evalc(polar(2,Pi/4))` should give `1.41421 + 1.41421 I`. Note `Pi` is the predefined name for the internal value of π . The inverse conversion uses the `convert` function, i.e., `convert(2+3*I, polar)`; might return

```
(1/2)
polar(13      , arctan(3/2))
```

As in the TI-85/86 case, if given w as one, say, third root of z you can find the others by

```
w*polar(1, 2*Pi/3)    w*polar(1, 4*Pi/3)
```

Incidentally, I should mention that MAPLE works in “adjustable accuracy” which means that you can have as many digits calculated

and printed out as you wish. If too many digits are being printed out for your tastes, use `Digits:=5;` to set the output to 5 digits. On the other hand if you *want* to see 40 digits, use `Digits:= 40;` and you will get them, and they will be mostly all accurate.