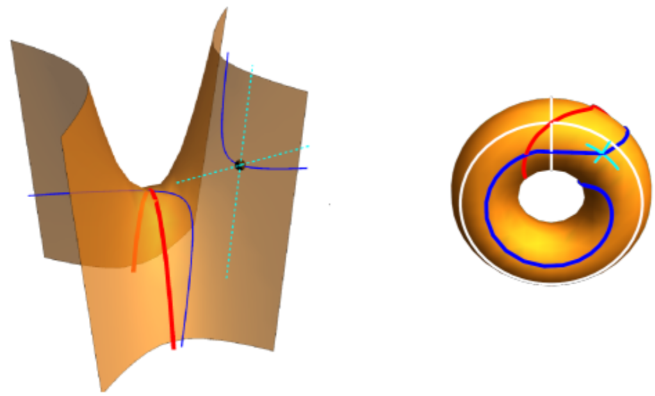

Front Graphic

The left hand graphic depicts the saddle surface, sometimes known as a hyperbolic paraboloid but it is a hyperboloid given by the equation $z = x\,y$. The intersection of this surface with the plane $x + y = 0$ is the parabola shown in red which has an infinite point at (0,0,-1,0). The saddle surface, as a hyperboloid, has many projective symmetries, that is invertible projective linear transformations which take this surface onto itself. One of these can be given by the 4×4 matrix for homogeneous coordinates $\{x, y, z, w\}$.

In[◦]:= sr3 =

$$\begin{pmatrix} 4.297877056362575` & 1.1516126864206027` & 2.224744871391589` & 2.224744871391589` \\ 0.11633650601052004` & 0.4341737512063021` & 0.22474487139158894` & 0.22474487139158894` \\ 0.7071067811865475` & 0.7071067811865475` & 1.3660254037844386` & 0.3660254037844386` \\ 0.7071067811865475` & 0.7071067811865475` & 0.3660254037844386` & 1.3660254037844386` \end{pmatrix};$$

This is given by its machine number equivalent here but in fact is exact, expressible by complicated entries using $\sqrt{2}$, $\sqrt{3}$. You may, for example, recognize the first two entries in the bottom two rows.

The blue curve is the image of the red parabola under this symmetry. The black point on this curve is the image of the invisible, (AKA infinite, ideal) point of the parabola. The two dashed lines form the image of the invisible curve. Specifically if one writes the homogeneous equation of this hyperboloid $z\,w - x\,y = 0$ the preimage of these lines form the intersection of this surface by the plane $w = 0$. Thus from this one picture we can see what the entire hyperboloid looks like. A major takeaway from this picture is that this saddle surface (which could be called a parabolic hyperboloid) is tangent to the plane at infinity. In fact, in affine space this is what distinguishes the parabolic hyperboloid from other hyperboloids. For more information see Chapter 2 Section 9 of my Surface Story https://barryhdayton.space/SurfaceBook/surfdex.html

As surfaces in real projective three space hyperboloids are topologically a torus. A homeomorphic image of the saddle surface is given in the right graphic which depicts the exact same situation. The singular white curve is the infinite curve of the saddle surface, that is, its intersection with the infinite plane. The intersection with the blue curve visible in this projection is actually below the left graphic, not the intersection near the point {0,0,0} of the saddle surface which is on the back side of the torus. Note how the red parabola goes through the singular point of the infinite curve. The cyan colored curves are the images under this symmetry of that intersection. More information on this will appear in a future Chapter 5 of the Surface Story.