Research Report
Analysis of Some Bertini Experiments
on Numerical Realizations of unions of Lines

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August 13, 2009

1 Introduction

In [3, 2] I considered certain unions of lines related to the classical Schäfli double-six, constructed numerically generic realizations and calculated the ideals of these realizations. In general these did not turn out to be ideal theoretic complete intersections, in one case, a double-four configuration of lines in $\mathbb{P}^3$, 7 generators were needed. A question was *how many generators are needed to obtain this configuration as a set theoretic intersection*?

I attempted to answer this question using Bertini [1]. I gave results of the Bertini Experiments in Table 6 of each paper. At the time I did not know how to independently check the results and simply reported on the experiments. Thanks to suggestions by Dan Bates and a later version of Bertini, Table 6 in [3] is more accurate than Table 6 of the earlier [2]. In this report I now explain how to check Table 6 using Mathematica and give the correct results, which match Table 6 in [3] very closely. I am now also confident of my count of the minimal number of generators needed to obtain a set theoretic intersection for each configuration tested.

Numerical irreducible decomposition is inherently unstable as a small perturbation of the system can change the nature of the solution. If the reducible algebraic set is a complete intersection typically the number of components may vary but the dimensions and total degree will generally stay the same. But as shown in [3], many reducible algebraic sets are non-complete intersections and then a small perturbation of the over determined system can drastically change dimension and degree as well.

For example consider the 3-3 configuration in Table 3 of [3] which is a complete intersection of a quadratic and cubic in $\mathbb{P}^3$ (See attached Appendices A,B). The irreducible decomposition given by Bertini 1.1, using fixed looser tolerances than the default, consists of 6 components of degree 1. Adding an arbitrary $.000337w^2$ to the quadratic equation changes the decomposition to 1 degree 2 and 1 degree 4 component. But if the perturbation is by adding $.0337w^2$ to the quadratic then one gets one component of degree 6. Contrast, however, the 3-2 configuration which is obtained as the solution of the two equations above plus another cubic. Without pertubation the Bertini solution is 5 components of dimension 1 degree 1 with a small (1-4 depending on run) number of extraneous components of dimension 0, adding the $.00337w^2$ to the first equation only gives 15 components of dimension 0 degree 1, i.e. 15 simple points.

Given this instability the Bertini results reported in Table 6 of [3] should be considered impressive. Still, I was concerned about the extraneous points returned not only because most of these were not actual solutions, but more importantly because I was not confident of my answers to the set theoretic intersection question.

Details of the examples and computations in this report are included in an attached appendix in text file format so that the reader may easily import data into Mathematica, Maple, Bertini, etc. if he wishes to do so.

2 Two theorems about the Hilbert Polynomial

Unlike the numerical irreducible decomposition, the Hilbert polynomial of a system, as calculated in [3] using approximate linear algebra, is relatively stable. By setting a loose tolerance the rank of the Macaulay arrays are lowered giving a larger Hilbert function and polynomial. In general one can expect to get the largest Hilbert polynomial of any nearby system. For example the 3-2 configuration above perturbed by $.00337w^2$ still gives the Hilbert function of the original 3-2 configuration when calculated using a tolerance of $\varepsilon = .002$.

There are several ways in which the Hilbert polynomial can detect the existence or non-existence of change in the algebraic set by adding or deleting ideal generators. I discuss this in this section.

As in [3] I will write $HF(d) = HF_{\mathcal{I}}(d) = \binom{d+s-1}{s} - \dim_{\mathbb{C}} \mathcal{I}_{(d)}$ where $\mathcal{I}_{(d)}$ is the degree d homogeneous part of $\mathcal{I}$ and the dimension is calculated relative to a tolerance ϵ, specifically this calculation is done using the Homogeneous Macaulay Array of [3] where all singular values less than ϵ are discarded. For large $d, HF(d)$ is a rational polynomial in d called the Hilbert Polynomial $HP(d) = HP_{\mathcal{I}}(d)$.

Theorem 1 *Suppose $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathbb{C}[x_0, \ldots, x_s]$ are homogeneous ideals. Suppose $\mathcal{I}, \mathcal{J}$ are each generated by a system of generators of degree n or less. If $H_{\mathcal{I}}(n) = H_{\mathcal{J}}(n)$ then $\mathcal{J} \subseteq \sqrt{\mathcal{I}}$ so $V(\mathcal{I}) = V(\mathcal{J})$, in particular if $\mathcal{I}, \mathcal{J}$ have the same Hilbert Polynomials then $V(\mathcal{I}) = V(\mathcal{J})$.*

Proof: First note that since $\mathcal{I} \subseteq \mathcal{J}$ the equality $H_{\mathcal{I}}(n) = H_{\mathcal{J}}(n)$ implies $\mathcal{I}_{(n)} = \mathcal{J}_{(n)}$. But then $\mathcal{I}_{(k)} = \mathcal{J}_{(k)}$ for all $k > n$ since there are no new generators of degree greater than n. So if $f \in \mathcal{J}$ there is an m so that $k = \deg(f^m) \geq n$ and thus $f^m \in \mathcal{I}$ hence $f \in \sqrt{\mathcal{I}}$. The conclusion $V(\mathcal{I}) = V(\mathcal{J})$ then follows. □

It should be noted that the converse is not true as the Hilbert Polynomial may in some cases be affected by the multiplicity of a component.

Theorem 2 *Suppose $\mathcal{J}$ is a homogeneous ideal in $\mathbb{C}[x_0, \ldots, x_s]$ and $\mathcal{K}$ is the homogeneous ideal of an simple point in $\mathbb{P}^s$ not contained in $V(\mathcal{J})$.*

i) $HP_{\mathcal{K}\cap\mathcal{J}} = HP_{\mathcal{J}} + 1$, *i.e.* $HF_{\mathcal{K}\cap\mathcal{J}}(d) = HF_{\mathcal{J}}(d) + 1$ *for large* d.

ii) *if* $V(\mathcal{I}) = V(\mathcal{J}) \cup \{P_1, \ldots, P_k\}$ *where* $P_1, \ldots, P_k$ *are distinct simple points not contained in* $V(\mathcal{J})$ *then* $HP_{\mathcal{I}} = HP_{\mathcal{J}} + k$.

Proof: For i) note that $V(\mathcal{K}) \cap V(\mathcal{J}) = \emptyset$ so $\sqrt{V(\mathcal{K}) + V(\mathcal{J})} = \mathfrak{m}$ the irrelevant ideal. Since $\mathcal{K}$ is the ideal of a simple point $HP_{\mathcal{K}} = 1$. But by [4, Prop. 5.4.16] $HP_{\mathcal{K}+\mathcal{J}} = 0$ and

$$HP_{\mathcal{K}\cap\mathcal{J}}(d) = HP_{\mathcal{K}}(d) + HP_{\mathcal{J}}(d) - HP_{\mathcal{I}+cJ}(d) = HP_{\mathcal{K}}(d) + HP_{\mathcal{J}}(d) = HP_{\mathcal{J}}(d) + 1$$

Part ii) follows from i) by induction. □

Again I note that the converse is not true as there are other ways to increase a Hilbert polynomial by a constant k. But combining the two theorems gives the following corollary

Corollary 1 *Suppose $\mathcal{I} \subseteq \mathcal{J}$ are homogeneous ideals and $HP_{\mathcal{I}} = HP_{\mathcal{J}} + k$ for some integer $k \geq 0$. If there are k known points $P_i, i = 1, \ldots, k$, with $P_i \in V(\mathcal{I})$ but $P_i \notin V(\mathcal{J})$ then $V(\mathcal{I}) = V(\mathcal{J}) \cup \{P_1, \ldots, P_k\}$*

Proof: Since $V(\mathcal{J}) \cup \{P_1, \ldots, P_k\} \subseteq V(\mathcal{I})$ then

$$\mathcal{I} \subseteq \sqrt{\mathcal{I}} = I(V(\mathcal{I})) \subseteq I(V(\mathcal{J}) \cup \{P_1, \ldots, P_k\}) \tag{1}$$

By the hypotheses and Theorem 2ii) we have an inclusion of two ideals with the same Hilbert polynomial so applying $V(\cdot)$ to (1) by Theorem 1

$$V(\mathcal{I}) = V\big(I\big(V(\mathcal{J}) \cup \{P_1, \ldots, P_k\}\big)\big) = V(\mathcal{J}) \cup \{P_1, \ldots, P_k\}$$

□

In the numerical case the Hilbert functions are calculated relative to a tolerance and whether a point is actually in $V(\mathcal{I})$ may also be subject to numerical considerations. But, as mentioned above, loosening the tolerance tend to increase the Hilbert function, so if the calculation of $HF_{\mathcal{I}}$ is done with a relatively loose tolerance and $HF_{\mathcal{J}}$ to a tighter tolerance one can accept the corollary as true with a high level of confidence.

Example 1: A radical ideal $\mathcal{K} = \langle c_1, c_2, c_3, c_4, c_5\rangle$ is constructed using the ideals of random points and lines in $\mathbb{P}^3$ and Algorithm 2 of [2, 3]. It happens that $c_1, \ldots, c_5$ are quadrics in $\mathbb{R}[x, y, z, w]$. Next $b_1, \ldots, b_4$ are given by random linear combinations of $c_1, \ldots, c_5$ and $\mathcal{J} = \langle b_1, b_2, b_3, b_4\rangle$. Finally a_1, a_2, a_3 are random linear combinations of $b_1, \ldots, b_4$ and $\mathcal{I} = \langle a_1, a_2, a_3\rangle$. So $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{K}$. See Appendix C for the specific polynomials.

Using the notation of [2, 3] the three Hilbert Functions are

$$HF_{\mathcal{I}} = 1, 4, 7, 8, 9, 10, 11 \to d + 5$$
$$HF_{\mathcal{J}} = 1, 4, 6, 6, 7, 8, 9 \to d + 3$$
$$HF_{\mathcal{K}} = 1, 4, 5, 6, 7, 8, 9 \to d + 3$$

where $HF_{\mathcal{I}}$ is calculated with a tolerance of 10^{-5} and the others to a tolerance of 10^{-10}. Since $\mathcal{J} \subseteq \mathcal{K}$ and appear to have the same Hilbert Polynomial we can conclude from Theorem 1 that $V(\mathcal{J}) = V(\mathcal{K})$. We would like to conclude that $V(\mathcal{I})$ consists of $V(\mathcal{K})$ with two additional points, by Corollary 1 given the above Hilbert functions it is enough to exhibit two points in $V(\mathcal{I})$ that are not in $V(\mathcal{K}) = V(\mathcal{J})$ We augment the system given by a_1, a_2, a_3 with a random linear equation *ell*, then using MATHEMATICA'S `NSolve` the small numerical inconsistencies leads MATHEMATICA to believe the ideal $\langle a_1, a_2, a_3, \ell\rangle$ is zero dimensional and returns 8 solution points in $V(\mathcal{I})$. Two of these are not in $V(\mathcal{J}) = V(\mathcal{K})$ which is all we needed to show.

Because of the random construction of $\mathcal{I}$ from $\mathcal{J}$ from $\mathcal{K}$ it is unlikely that there are any accidental relations among the a_i, b_i. This suggests that the algebraic set $V(\mathcal{K})$ is not a set theoretic intersection of three equations but is a set theoretic intersection of 4 equations, but, since $HF_{\mathcal{J}}(3) \neq HF_{\mathcal{K}}(3)$ this variety is not an ideal theoretic intersection of 4 equations.

A more complicated example of this is the Double-four not contained in a cubic. See below and Appendix D for details.

3 Table 6 of [3] revisited

For the reader who may not have [3] handy, I include the original Table 6 as Table 1. The lists of generators in the first column have been corrected, but otherwise this table is unchanged from [3].

The constructions in [2, 3] were done line by line using Algorithm 2 of [2, 3] which numerically finds a reduced basis for the intersection of two ideals. Since the ideals of the original lines were given by two homogeneous linear equations these ideals of the lines are clearly radical, in fact, prime ideals in $\mathbb{C}[x_0, \ldots, x_3]$. Further it is elementary that the intersection of radical ideals is radical, so the final ideal $\mathcal{I}$ of a configuration is radical. However this is an intersection of the ideals in the construction, so the final generators may not have been in any of the previous ideals. So sub-ideals of $\mathcal{I}$ generated by a subset of the basis may not be radical. In fact, we will see below that they are often not radical. Given the results of the previous section I redo Table 6 of [3] giving the Hilbert Functions and polynomials rather than "other points".

Here type refers to *set theoretic intersection* (STI) where $V(\mathcal{I})$ is the desired figure but the ideal is not radical and *ideal theoretic intersection* (ITI) where the ideal is radical, i.e. $\mathcal{I} = I(V(\mathcal{I}))$. The subscripts on the generators refer to the total degree of the generator so, for example s_4 is the generator of the double-six of total degree 4. The convention on Hilbert functions is as in [3] and above where the Hilbert polynomial is given after the Hilbert function begins to agree with it.

The results for the first four configurations were correct in the original Table 6, so long as points where not counted unless they passed the *membership test*, i.e. the points in the last column of the original Table 6 were rejected. In particular the 4 additional points found in the algebraic set determined by generators a_4, b_4, c_4 of the double-five not in a cubic were exactly those returned by the original run and membership test by BERTINI. I speculate that BERTINI

Table 1: Table 6 of [3]

Generators	Lines	other 1-dim	isolated points	other points
Double-six, generators s_3, s_4				
s_3, s_4	12		0	0
Double-five subvariety, generators s_3, s_4, a_5, b_5				
s_3, a_5	10	1 deg 5	0	0
s_3, s_4, a_5	10		0	12
s_3, s_4, a_5, b_5	10		0	21
Double-five not subvariety, generators a_4, b_4, c_4, d_4, e_4				
a_4, b_4	10	1 deg 6	0	0
a_4, b_4, c_4	10		4	6
a_4, b_4, c_4, d_4	10		0	4
a_4, b_4, c_4, d_4, e_4	10		0	3
Double-four subvariety, generators s_3, s_4, a_4, b_4				
s_3, a_4	8	2 deg 2	0	0
s_3, s_4, a_4	8		0	5
s_3, s_4, a_4, b_4	8		0	8
Double-four not subvariety, generators $a_4, \ldots, g_4$				
a_4, b_4	8	1 deg 8	0	0
a_4, b_4, c_4	8		7	5
a_4, b_4, c_4, d_4	8		0	2
a_4, b_4, c_4, d_4, e_4	8		0	1
$a_4, \ldots, g_4$	8		0	0
Double-three subvariety, generators s_3, a_3, s_4, a_4, b_4				
s_3, a_3	9		0	0
s_3, a_3, s_4	6		1	11
s_3, a_3, s_4, a_4	6		0	8
s_3, a_3, s_4, a_4, b_4	6		0	7
Double-two subvariety, generators a_2, s_3, a_3, s_4				
a_2, s_3	4	1 deg 2	0	0
a_2, a_3, s_3	5		0	2
a_2, a_3, s_3, s_4	4		0	18

Table 2: Table 6 of [3] Revised

Generators	Lines	other	Hilbert function	type
Double-six, generators s_3, s_4				
s_3, s_4	12		$1, 4, 10, 19, 30, 42 \to 12d - 18$	ITI
Double-five subvariety, generators s_3, s_4, a_5, b_5				
s_3, a_5	10	dim 1 deg 5	$1, 4, 10, 19, 31, 45 \to 15d - 30$	
s_3, s_4, a_5	10		$1, 4, 10, 19, 30, 41, 50 \to 10d - 10$	STI
s_3, s_4, a_5, b_5	10		$1, 4, 10, 19, 30, 40 \to 10d - 10$	ITI
Double-five not subvariety, generators a_4, b_4, c_4, d_4, e_4				
a_4, b_4	10	dim 1 deg 6	$1, 4, 10, 20, 33, 48 \to 16d - 32$	
a_4, b_4, c_4	10	4 points	$1, 4, 10, 20, 32, 44 \to 10d - 6$	
a_4, b_4, c_4, d_4	10		$1, 4, 10, 20, 31, 40 \to 10d - 10$	STI
a_4, b_4, c_4, d_4, e_4	10		$1, 4, 10, 20, 30, 40 \to 10d - 10$	ITI
Double-four subvariety, generators s_3, s_4, a_4, b_4				
s_3, a_4	8	2 dim 1 deg 2	$1, 4, 10, 19, 30, 42 \to 12d - 18$	
s_3, s_4, a_4	8		$1, 4, 10, 19, 29, 38, 45, 52 \to 8d - 4$	STI
$s3, s_4, a_4, b_4$	8		$1, 4, 10, 19, 28, 36 \to 8d - 4$	ITI
Double-four not subvariety, generators $a_4, \ldots, g_4$				
a_4, b_4	8	dim 1 deg 8	$1, 4, 10, 20, 33, 48 \to 16d - 32$	
a_4, b_4, c_4	8	8 points	$1, 4, 10, 20, 32, 44, 54, 69 \to 8d + 4$	
a_4, b_4, c_4, d_4	8		$1, 4, 10, 20, 31, 40, 44 \to 8d - 4$	STI
a_4, b_4, c_4, d_4, e_4	8		$1, 4, 10, 20, 30, 36, 44 \to 8d - 4$	STI
$a_4, \ldots, f_4$	8		$1, 4, 10, 20, 29, 36 \to 8d - 4$	STI
$a_4, \ldots, g_4$	8		$1, 4, 10, 20, 28, 36 \to 8d - 4$	ITI
Double-three subvariety, generators s_3, a_3, s_4, a_4, b_4				
s_3, a_3	9		$1, 4, 10, 18, 27, 36 \to 9d - 9$	
s_3, a_3, s_4	6		$1, 4, 10, 18, 26, 32, 36 \to 6d$	STI
s_3, a_3, s_4, a_4	6		$1, 4, 10, 18, 25, 30, 36 \to 6d$	STI
s_3, a_3, s_4, a_4, b_4	6		$1, 4, 10, 18, 24, 30, 36 \to 6d$	ITI
Double-two subvariety, generators a_2, s_3, a_3, s_4				
a_2, s_3	4	dim 1 deg 2	$1, 4, 9, 15, 21, 27 \to 6d - 3$	
a_2, s_3, s_4	4		$1, 4, 9, 15, 20, 23, 26 \to 4d + 2$	STI
a_2, a_3, s_3, s_4	4		$1, 4, 9, 14, 18, 22, 26 \to 4d + 2$	ITI

had an easier time with this configuration because for this one configuration the equations $a_4, \ldots, e_4$ all had complex, not real, coefficients.

For the double-four not in a cubic the three generator set had 8, not 7, extra points. Six of these 8 points were among the 7 points reported in the original Table 6, one was not a correct isolated point, and two additional points were found by an additional BERTINI run with the same data. See Appendix D for the equations and solutions. It should be noted that since Newton iteration converges well to a simple isolated point, before using the membership test the correct isolated zeros were among those points returned by the original BERTINI run with "good" condition numbers. Thus the points with "bad" condition numbers did not need to be tested. I also note that because of the randomness in the construction and numerical methods used, lots of singular value decompositions, the generators $a_4, \ldots, g_4$ were random among all possible generators and so using them in a different order would have given similar results. But once fixing $\mathcal{I} = \langle a_4, b_4, c_4, d_4 \rangle$ as an ideal for a STI of the double-four then e_4, f_4, g_4 are in the the radical of $\mathcal{I}$ and needed to generate this radical to get the ITI.

The double-three subvariety was correct in Table 6 except for the isolated point found for generators $\{a_3, s_3, s_4\}$. The Hilbert function calculation shows that the radical of the ideal generated by these generators is the radical ideal of the double-three so there are no points not on the double-three. It should be noted that if one starts with generators $\{s_3, s_4, a_4\}$ then it is geometrically clear that there will be at least 6 isolated solutions since each of lines 4,5,6,10,11,12 of the double-six meet the double-three in 3 points but must meet a_4 in 4 points, so one must not be on the double-three. Since all points of the double-six are zeros of s_3, s_4 these 6 distinct points satisfy the three equations. By Corollary 1 a Hilbert polynomial calculation shows that these are the only isolated points of this system.

The double-two was correct in Table 6 except for an error in properly identifying the generators in the first and third calculation. The generating set $\{a_3, s_3\}$ originally listed in Table 6 of [3] would have a solution of total degree 9, 5 lines and a degree 4 curve. The correct generators for the answer given in Table 6 are listed here. The middle calculation has been replaced by the generating set $\{a_2, s_3, s_4\}$ to show, by Theorem 1, that this configuration also is a STI of a proper subset of the the basis of the ITI, a_3 is evidently in the radical of $\langle a_2, s_3, s_4 \rangle$.

4 Comments

The BERTINI results, after membership testing, given in Table 6 of [3] were correct with only two exceptions and then off by only one point. I show here that additional testing can be done and one can identify exactly which of the many points returned are actually isolated points of the system. It seems that the many extraneous points found are artifacts of the fact that the generic "square system", i.e. n homogeneous equations in $n + 1$ variables, have many isolated solutions. Thus even the small inaccuracies in my equations caused BERTINI to find many of these. It does not seem to me now that changing tolerances in BERTINI will settle these questions since different tolerances and even different runs will produce varying numbers of points, some of which will be extraneous and some missing, so there is no internal way in BERTINI to know which calculation is correct.

References

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