Algebraic Foundation of Local Multiplicity

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based on joint work with
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1 Introduction

In a preliminary version of [4] we wished to give a complete discussion of the underlying theory that would be accessible to a typical reader of *Mathematics of Computation*. The editors of our manuscript decided that this material was well known and should not re-appear in a journal article. Consequently the published version of [4] will contain only a shortened version of the original material.

In my recent preprint [2] I have made a careful distinction between the *local duality theory* vs. the *global duality theory*. What was well known was the global duality theory whereas the theory we presented is the local duality theory which is quite different and not well known. In particular we know of no convenient reference. Since the global duality theory has been restricted largely to zero-dimensional rings, i.e. finitely many solution points, the local theory could be considered a special case. But although our main interest in [4] was isolated zeros we did not restrict to the case where the ring in question was zero-dimensional. So our local theory is not a direct consequence of the global theory and the distinction becomes more obvious in positive dimensions.

Therefore I feel that it is worthwhile to at least post this complete version of the theory, at this point I do not intend to attempt to publish it. In preparing [4] it was my assignment to write the first draft of the material included here, and I take full responsibility for the revisions in this posting to make it as self contained as possible. However I owe a big debt to my co-authors of [4] for encouraging me to write this down, for editing my original, and in particular to T. Y. Li for forcing me to get this correct and complete modulo easily accessible references.

The reader needs to be familiar with the concepts of Macaulay Matrices and the dual space of differential forms as outlined in any one of [3, 4, 2]. In particular in the first two the *multiplicity* of an isolated zero is defined by

Definition 1: Let $F = \{f_1, \ldots, f_t\}$ be a system of functions having derivatives of order $\gamma \geq 1$ at the zero $\hat{\mathbf{x}} \in \mathbb{C}^s$, assume $\mathcal{D}^{\gamma-1}_{\hat{\mathbf{x}}}(F) = \mathcal{D}^{\gamma}_{\hat{\mathbf{x}}}(F)$ where $\mathcal{D}^{\alpha}_{\hat{\mathbf{x}}}(F)$ is the space of local dual differential functionals at $\hat{\mathbf{x}}$ of order α or less. The *multiplicity of F at $\hat{\mathbf{x}}$* is the $\mathbb{C}$-dimension of $\mathcal{D}^{\gamma}_{\hat{\mathbf{x}}}(F)$.

2 Analytic Preliminaries

The main result of this section is a lemma which connects the analytic concept of an isolated zero of a system with the algebraic-geometric concept. We will also need a technical lemma.

Let $\mathbb{C}[x_1, \ldots, x_s]$ be the ring of polynomials in variables $x_1, \ldots, x_s$, while $\mathbb{C}\{x_1, \ldots, x_s\}$ denotes the ring of convergent power series centered at $\mathbf{0}$, that is each element $f \in \mathbb{C}\{x_1, \ldots, x_s\}$ converges in some open set $\mathcal{V}_f$ about $\mathbf{0}$, see [1, 10]. For a set $F = \{f_1, \ldots, f_t\}$ of analytic functions and a ring $\mathcal{R}$ of functions on $\mathcal{U}$, let $F\mathcal{R}$ denote the ideal

$$F\mathcal{R} \quad = \quad \big\{ f_1 g_1 + \cdots + f_t g_t \;\big|\; g_1, \ldots, g_t \in \mathcal{R} \big\}. \tag{1}$$

For an analytic function, define $\mathit{ord}(f) = m$ to be the smallest integer such that a term of the multivariate Taylor series of f at $\mathbf{0}$ of total degree m has a nonzero coefficient. Define

$$\mathit{jet}(f, k) \quad = \quad \sum_{|\alpha| \le k} f_\alpha \tag{2}$$

where for $\alpha = x_1^{j_1} \ldots x_s^{j_s}$, $|\alpha| = j_1 + \cdots + j_s$ is the total degree and f_α is the term $c_\alpha \alpha$ in a Taylor series expansion of f about $\mathbf{0}$. Note that jet is a linear map on $\mathcal{R}$ and that $\mathit{ord}(f - \mathit{jet}(f, m)) > m$. An *open polydisc* in $\mathbb{C}^s$ at a point $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$ is defined as

$$\Delta(\hat{\mathbf{x}}, \mathbf{r}) \quad = \quad \{ \mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{C}^s \;\big|\; |a_i - \hat{x}_s| < r_i, \ i = 1, \ldots, s \}$$

for an array $\mathbf{r} = [r_1, \ldots, r_s] \in \mathbb{N}_+^s$ of positive real numbers. From an analytic point of view an *isolated zero* is defined by

Definition 1 *A point* $\hat{\mathbf{x}}$ *is an* isolated zero *of a system* $F = \{f_1, \ldots, f_t\}$ *means that there is an open polydisc* $\Delta(\hat{\mathbf{x}}, \mathbf{r})$ *in which* $\hat{\mathbf{x}}$ *is the only zero of* F.

Lemma 1 *Let* $\mathcal{R}$ *be the ring of analytic functions on open set* $\mathcal{U} \subseteq \mathbb{C}^s$ *and assume* $\hat{\mathbf{x}} = \mathbf{0} \in \mathcal{U}$. *Let* $F = \{f_1, \ldots, f_t\} \subset \mathcal{R}$ *be a system of analytic functions with common zero* $\hat{\mathbf{x}}$. *Then the following are equivalent:*

(i) *The point* $\hat{\mathbf{x}} = \mathbf{0} \in \mathcal{U}$ *is an isolated zero of* F.

(ii) *For each* $j \in \{1, \ldots, s\}$ *there is an integer* e_j *such that* $x_j^{e_j} \in F\mathbb{C}\{x_1, \ldots, x_s\}$.

(iii) *For each* $j \in \{1, \ldots, s\}$ *there is an analytic function* $p_j \in \mathcal{R}$ *with* $\mathit{ord}(p_j) > e_j$ *and* $x_j^{e_j} + p_j \in F\mathbb{C}[x_1, \ldots, x_s]$.

Proof. We first prove (i) implies (ii) following Rükert's Nullstellensatz [10, Theorem 4.5.5]. Let $F^* = F\mathbb{C}\{x_1, \ldots, x_s\}$ be the ideal generated by F in $\mathbb{C}\{x_1, ..x_s\}$. Then loc F^* is the germ of the one element variety $\{\mathbf{0}\}$, using the notation in [10], By the Nullstellensatz, the radical of $F^* = \sqrt{F^*} =$ id loc F^* is the maximal ideal of $\mathbb{C}\{x_1, \ldots, x_s\}$ at $\mathbf{0}$. Notice that

$$\sqrt{F^*} \quad = \quad \{ f \in \mathbb{C}\{x_1, \ldots, x_s\} \;\big|\; f^k \in F^* \quad \text{for some integer} \ \ k > 0 \}.$$

But $x_1, \ldots, x_s$ are all in this maximal ideal so there exist integers $e_1, \ldots, e_s > 0$ such that $x_1^{e_1}, \ldots, x_s^{e_s} \in F^*$, proving the assertion (ii).

We then prove (ii) implies (iii). For a particular $j \in \{1, \ldots, s\}$ the assertion (i) implies that there exist $g_1, \ldots, g_t \in \mathbb{C}\{x_1, \ldots, x_s\}$ with

$$x_j^{e_j} = g_1 f_1 + \cdots + g_t f_t \tag{3}$$

Let $\tilde{g}_i = g_i - \mathit{jet}(g_i, e_j)$. Then $\mathit{ord}(\tilde{g}_i) > e_j$. Therefore, for each j, from (3)

$$\begin{aligned} x_j^{e_j} &= (\mathit{jet}(g_1, e_j) + \tilde{g}_1) f_1 + \cdots + (\mathit{jet}(g_t, e_j) + \tilde{g}_t) f_t \\ &= \big(\mathit{jet}(g_1, e_j) f_1 + \ldots \mathit{jet}(g_t, e_j) f_t\big) + \big(\tilde{g}_1 f_1 + \ldots \tilde{g}_t f_t\big) \end{aligned} \tag{4}$$

so

$$x_j^{e_j} + p_j = \mathit{jet}(g_1, e_j) f_1 + \cdots + \mathit{jet}(g_t, e_j) f_t \in F\mathbb{C}[x_1, \ldots, x_s] \tag{5}$$

where $p_j = -\big(\tilde{g}_1 f_1 + \cdots + \tilde{g}_t f_t\big)$. But then $\mathit{ord}(p_j) > e_j$ and the right hand side of (5) is in $F\mathcal{R}$ since the $\mathit{jet}(g_i, e_j)$ are polynomials and thus in $\mathcal{R}$. Moreover, since also $x_j^{e_j} \in \mathcal{R}$ it follows that actually $p_j \in \mathcal{R}$.

We now prove (iii) implies (i). Assume the assertion (iii) holds. Then by Schwarz's Lemma [10, Exercise 4, p. 35] for each $j = 1, \ldots, s$ there exists a constant K_j such that $|p_j(\mathbf{x})| \le K_j \|\mathbf{x}\|^{e_j+1}$ in a polydisc $\Delta(\mathbf{0}, [t_j, \ldots, t_j])$ of $\mathbf{0}$. Let $r_i = \min\{t_j, \frac{1}{2K_j}\}$, $\mathbf{r} = [r_1, \ldots, r_s]$ and let

$$\mathcal{V}_j = \Big\{ \mathbf{x} = (x_1, \ldots, x_s) \in \Delta(\mathbf{0}, \mathbf{r}) \setminus \{\mathbf{0}\} \ \Big| \ \max_{1 \le i \le s} |x_i| = |x_j| \Big\}$$

for $j = 1, \ldots, s$. Now in each $\mathcal{V}_j$ the inequality

$$|p_j(\mathbf{x})| \le K_i \|\mathbf{x}\|^{e_j+1} \le K_i |x_j|^{e_j+1} < K_i r_i |x_j|^{e_j}| = \frac{1}{2} |x_j^{e_j}| < |x_j^{e_j}|$$

and hence $x_j^{e_j} + p_j(\mathbf{x}) \neq 0$ in $\mathcal{V}_j$. But $x_j^{e_j} + p_j = g_1 f_1 + \cdots + g_t f_t$ for some $g_1, \ldots, g_t \in \mathcal{R}$, so for each $\mathbf{x} \in \Delta(\mathbf{0}, \mathbf{r}) \setminus \{\mathbf{0}\}$ some $f_i(\mathbf{x})$ must not vanish. Since we are using the infinity norm each $\mathbf{x} \in \Delta(\mathbf{0}, \mathbf{r})$ satisfies $\|\mathbf{x}\| = |x_j| < r_j$ for some j, so $\Delta(\mathbf{0}, \mathbf{r}) \setminus \{\mathbf{0}\} = \cup_{j=1}^{s} \mathcal{V}_j$ which proves the assertion (i). □

Remarks on Lemma 1: The form of (iii) implies that if $f_1, \ldots, f_t \subset \mathbb{C}[x_1, \ldots, x_s]$ are polynomials then also the p_j are polynomials in $\mathbb{C}[x_1, \ldots, x_s]$, so Lemma 1 specializes to a lemma about polynomials. Generally in algebra the practice is to give results over as general a field as possible so the ground field could be the rationals or a field of characteristic p and hence the analytic definition above does not apply. Condition (ii), which is equivalent to the local ring at $\hat{\mathbf{x}} = \mathbf{0}$ being finite dimensional over the ground field, is used as the working definition of an isolated zero so (i) $\Leftrightarrow$ (ii) by definition. For polynomials the difficulty in this lemma is using the analytic definition of isolated zero above. If it is known that the solution to the system F is finite, then by Hilbert's Nullstellensatz the ring $\mathbb{C}[x_1, \ldots, x_s]/F\mathbb{C}[x_1, \ldots, x_x]$ is finite dimensional over $\mathbb{C}$ and (ii) follows. But in the case where the system F also has positive dimensional components and $\hat{\mathbf{x}}$ is a multiple zero we know of no easier proof in the polynomial case than the one given here.

For the rest of this section $\mathfrak{M}_s^{\alpha}$, $\alpha = 0, 1, 2, \ldots$ will denote the ideal in $\mathbb{C}\{x_1, \ldots, x_s\}$ given by

$$\mathfrak{M}_s^{\alpha} = \{ f \in \mathbb{C}\{x_1, \ldots, x_s\} \Big| \mathit{ord}(f) \ge \alpha \}.$$

Note that $\mathfrak{M}_s^0 = \mathbb{C}\{x_1, \ldots, x_s\}$.

Lemma 2 *The ideal* $\mathfrak{M}_s^\alpha$ *is generated by the monomials* $\mathbf{x}^\mathbf{j}$ *with* $|\mathbf{j}| = \alpha$, *that is if* $f \in \mathfrak{M}_s^\alpha$ *then* $f = \sum_{|\mathbf{j}|=\alpha} \mathbf{x}^\mathbf{j} g_\mathbf{j}$ *where the* $\mathbf{x}^\mathbf{j}$ *and* $g_\mathbf{j}$ *are elements of* $\mathbb{C}\{x_1, \ldots, x_s\}$.

Proof. The proof goes by induction on s. If $s = 1$ then $f \in \mathfrak{M}_1^\alpha$ is of the form $f = \sum_{k \geq \alpha} a_k x_1^k = x_1^\alpha \sum_{j \geq 0} a_{j+\alpha} x_1^j$ so $\mathfrak{M}_1^\alpha$ is generated by x_1^α.

For $s > 1$ by the Weierstrass Division Theorem [10, Th. 3.3.5] $h = x_s^\alpha$ is a Weierstrass polynomial for x_s so if $f \in \mathfrak{M}_s^\alpha$ then $f = x_s^\alpha g + q$ where $g \in \mathbb{C}\{x_1, \ldots, x_s\}$ and q is a polynomial in x_s of degree $\alpha - 1$ or less in $\mathbb{C}\{x_1, \ldots, x_{s-1}\}[x_s]$. This means $q = x_s^{\alpha-1} q_1 + \cdots + x_s q_{\alpha-1} + q_\alpha$ where $q_k \in \mathbb{C}\{x_1, \ldots, x_{s-1}\}$. Since f and $x^\alpha g$ are in $\mathfrak{M}_s^\alpha$, it follows that $\mathit{ord}(q) \geq \alpha$ so $q_k \in \mathfrak{M}_{s-1}^k$. By induction each term of q is a sum of monomials in $\mathbb{C}\{x_1, \ldots, x_{s-1}\}$ of degree α times an element of $\mathbb{C}\{x_1, \ldots, x_{s-1}\}$ so f is of the desired form. □

Remark on Lemma 2 This lemma is also needed in the polynomial case if one is to use $\mathbb{C}[[x_1, \ldots, x_s]]$ rather than $\mathbb{C}[x_1, \ldots, x_s]_{\langle x_1, \ldots, x_s \rangle}$ as the local ring, see [1]. In [11] an algebraic proof of the Weierstrass preparation theorem is given for $\mathbb{C}[[x_1, \ldots, x_s]]$ from which the analog of this lemma can be deduced.

A consequence of lemmas 1 and 2 is that if $\hat{\mathbf{x}} = \mathbf{0}$ is an isolated zero of an analytic system $F = \{f_1, \ldots, f_t\}$ then $\mathbb{C}\{x_1, \ldots, x_s\}/F\mathbb{C}\{x_1, \ldots, x_s\}$ has finite dimension as a $\mathbb{C}$-algebra, see Theorem 1 part (ii) below. We will also need to refer to a lemma from [8] which uses this fact to define multiplicity as this dimension. For the convenience of the reader we paraphrase a special case of that lemma here using our notation.

Lemma 3 (Local Extension Lemma) [8, Lemma 6] *Let* $\tilde{F} = [\tilde{f}_1, \ldots, \tilde{f}_s]^\top : \mathbb{C}^s \times \mathbb{C}^m \to C^s$ *be a system of* s *holomorphic functions. Let* $\hat{\mathbf{x}}$ *be an isolated solution of* $\tilde{F}(\mathbf{x}, \hat{\mathbf{y}}) = \mathbf{0}$ *for a fixed* $\hat{\mathbf{y}}$. *Let* $m = \mathit{dim}_\mathbb{C}\Big(\mathbb{C}\{x_1, \ldots, x_s\}/\tilde{F}\mathbb{C}\{x_1, \ldots, x_s\}\Big)$. *Then there are open neighborhoods* $\mathcal{U}$ *of* $\hat{\mathbf{x}}$, $\mathcal{V}$ *of* $\hat{\mathbf{y}}$ *such that for any* $\mathbf{y} \in \mathcal{V}$ *there exist* m *isolated solutions* $\mathbf{x}$ *(counting multiplicities) of* $\tilde{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ *on* $\mathcal{U}$.

3 Equivalence of Hilbert Functions

Familiarity with the concept of duality in the ring setting and Macaulay Matrices is necessary for the next two sections. This material can be obtained from any of [3, 4, 2], note we are using S_α for the Macaulay matrix of order α and assume the zero of interest is the origin.

A major tool in commutative algebra is the *Hilbert Function.* See [9] for a classical discussion and [7] for an extensive and up to date discussion on the many versions of Hilbert Functions. Here we define a Hilbert function by

$$\begin{cases} \mathrm{h}(0) &= \mathit{dim}\Big(\mathcal{D}_{\hat{\mathbf{x}}}^0(F)\Big) \equiv 1 \\ \mathrm{h}(\alpha) &= \mathit{dim}\Big(\mathcal{D}_{\hat{\mathbf{x}}}^\alpha(F)\Big) - \mathit{dim}\Big(\mathcal{D}_{\hat{\mathbf{x}}}^{\alpha-1}(F)\Big) \quad \text{for } \alpha \in \{1, 2, \ldots\}, \end{cases}$$

and showed in [3, Theorem 1] and [4, Corollary 1]

$$\mathrm{h}(\alpha) = \mathit{nullity}\,(S_\alpha) - \mathit{nullity}\,(S_{\alpha-1}) \tag{6}$$

for $\alpha = 1, 2, \cdots$. Now we define a new Hilbert function $\mathrm{H}(\alpha)$ and show that it agrees with $\mathrm{h}(\alpha)$.

We will now hold the number of variables s fixed, so we drop the subscript s and just write $\mathfrak{M}^\alpha$ instead of $\mathfrak{M}_s^\alpha$. By Lemma 2 above each of the ideals $\mathfrak{M}^\alpha$ are finitely generated, in fact generated by the $\binom{s+\alpha-1}{s-1}$ monomials $\mathbf{x}^\mathbf{j}$ of total degree α. Note that the projection $\rho_\alpha : \mathbb{C}\{x_1, \ldots, x_s\} \to \mathbb{C}\{x_1, \ldots, x_s\}/\mathfrak{M}^\alpha$ satisfies $\rho_{\alpha+1}(f) = \rho_{\alpha+1}(\mathit{jet}(f, \alpha))$ from which it easily follows that $\mathfrak{M}^\alpha/\mathfrak{M}^{\alpha+1}$ has a basis consisting of the monomials $\mathbf{x}^\mathbf{j}$ of total degree α and so

$$\mathit{dim}\Big(\mathfrak{M}^\alpha/\mathfrak{M}^{\alpha+1}\Big) \quad = \quad \binom{\alpha + s - 1}{s - 1} \tag{7}$$

as a $\mathbb{C}$-vector space.

Now given an analytic system $F = \{f_1, \ldots, f_t\}$ we give a filtration $\mathfrak{m}^\alpha$

$$\mathcal{A} = \mathfrak{m}^0 \supseteq \mathfrak{m}^1 \supseteq \mathfrak{m}^2 \supseteq \ldots$$

on $\mathcal{A} = \mathbb{C}\{x_1, \ldots, x_s\}/F\mathbb{C}\{x_1, \ldots, x_s\}$ by defining $\mathfrak{m}^\alpha$ to be the image of $\mathfrak{M}^\alpha$ in $\mathcal{A}$. It is not hard to see that $\mathfrak{m}^\alpha \approx \mathfrak{M}^\alpha/(\mathfrak{M}^\alpha \cap F\mathbb{C}\{x_1, \ldots, x_s\})$. Further $\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}$ is a quotient of $\mathfrak{M}^\alpha/\mathfrak{M}^{\alpha+1}$ and so is a $\mathbb{C}$-vector space of dimension less than or equal to $\binom{\alpha+s-1}{s-1}$.

We define the Hilbert function $\mathrm{H}(\alpha)$ by

$$\begin{cases} \mathrm{H}(0) & = & \mathit{dim}\Big(\mathcal{A}/\mathfrak{m}^1\Big) \equiv 1 \\ \mathrm{H}(\alpha) & = & \mathit{dim}\Big(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}\Big) \quad \text{for} \ \ \alpha \in \{1, 2, \ldots\}. \end{cases} \tag{8}$$

We proceed to establish the following lemma.

Lemma 4 *Let* $F = \{f_1, \ldots, f_t\}$ *be a system of analytic equations on an open subset of* $\mathbb{C}^s$ *containing* $\hat{\mathbf{x}} = \mathbf{0}$ *such that* $\hat{\mathbf{x}}$ *is a common zero of the system. Then*

$$H(\alpha) \quad = \quad h(\alpha)$$

for all integers $\alpha = 0, 1, 2, \ldots$.

Proof. Let $\mathcal{I} = F\mathbb{C}\{x_1, \ldots, x_s\}$ then $\mathcal{A} = \mathbb{C}\{x_1, \ldots, x_s\}/\mathcal{I}$ as above. For $g \in \mathbb{C}\{x_1, \ldots, x_s\}$ we write $g = g^{(0)} + g^{(1)} + g^{(2)} + \cdots$ where $g^{(\lambda)}$ is the sum of all terms of total degree λ. We define the function In_α by

$$\mathit{In}_\alpha(g) \quad = \quad \begin{cases} g^{(\alpha)} & \text{if} \ \ g^{(\lambda)} = 0 \ \ \text{for} \ \ \lambda < \alpha \ \ \text{and} \ \ g^{(\alpha)} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For the ideal $\mathcal{J}$ we write $\mathit{In}_\alpha(\mathcal{J}) = \{\mathit{In}_\alpha(f) \big| f \in \mathcal{J}\}$.

We then obtain a commutative diagram

$$
\begin{array}{ccccccccc}
 & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{0} & \longrightarrow & \mathfrak{M}^{\alpha+1}\cap\mathcal{I} & \longrightarrow & \mathfrak{M}^{\alpha}\cap\mathcal{I} & \xrightarrow{In_\alpha} & In_\alpha(\mathcal{I}) & \longrightarrow & \mathbf{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{0} & \longrightarrow & \mathfrak{M}^{\alpha+1} & \longrightarrow & \mathfrak{M}^{\alpha} & \xrightarrow{In_\alpha} & In_\alpha(\mathfrak{M}^\alpha) & \longrightarrow & \mathbf{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{0} & \longrightarrow & \mathfrak{m}^{\alpha+1} & \longrightarrow & \mathfrak{m}^{\alpha} & \longrightarrow & \mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1} & \longrightarrow & \mathbf{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & &
\end{array}
$$

where the unmarked linear maps are the obvious inclusions or projections except for the last column. There $In_\alpha(\mathcal{I})$ consists of forms of degree α which are initial forms of elemensts of $\mathcal{I}$ and $In_\alpha(\mathfrak{M}^\alpha)$ is just the space of all forms of degree α so $In_\alpha(\mathcal{I}) \to In_\alpha(\mathfrak{M}^\alpha)$ is again an inclusion. The map $In_\alpha(\mathfrak{M}^\alpha) \to \mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}$ is induced by the projection $\mathfrak{M}^\alpha \to \mathfrak{m}^\alpha$, i.e. is the unique map making the bottom right square of the diagram commute.

The rows of this diagram are all exact (image in is kernel out) as are the first two columns. Exactness of the third column then follows from the 3×3-lemma of homological algebra. As a consequence of this we have

$$H(\alpha) \;=\; dim\big(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}\big) \;=\; dim_{\mathbb{C}}(In_\alpha(\mathfrak{M}^\alpha)) - dim_{\mathbb{C}}(In_\alpha(\mathcal{I})) \;=\; \binom{\alpha-1+s}{s-1} - dim_{\mathbb{C}}(In_\alpha(\mathcal{I})) \tag{9}$$

thus it is enough to calculate $dim_{\mathbb{C}}\big(In_\alpha(\mathcal{I})\big)$.

The proof will finish by calculating $dim\big(In_\alpha(\mathcal{I})\big)$ using the Macaulay matrix S_α. We will write $jet(\mathcal{I},\alpha)$ for the vector space spanned by $\{jet(g,\alpha) | g \in \mathcal{I}\}$. Since $\hat{\mathbf{x}} = \mathbf{0}$ the entry in row labeled $\mathbf{x}^{\mathbf{k}} f_i$ and column indexed $\mathbf{x}^j$ is just the coefficient of $\mathbf{x}^j$ of the polynomial $\mathbf{x}^{\mathbf{k}} f_i$ it is seen that $jet(\mathcal{I},\alpha)$ is just the rowspace of S_α. Now, in theory only, the reduced row echelon form algorithm will put S_α in echelon form by row operations. In particular S_α is row equivalent to a matrix with independent rows

$$S_\alpha \;\simeq\; A_\alpha \;=\; \left[\begin{array}{c|c} rowspace\ S_{\alpha-1} & B_\alpha \\ \hline \mathbf{0} & C_\alpha \end{array}\right] \tag{10}$$

Now each row of C_α corresponds to an element of $In_\alpha(\mathcal{I})$ by multiplying each entry by its column index and adding. These elements clearly form a basis for $In_\alpha(\mathcal{I})$.

The total number of rows in A_α is $rank(S_\alpha)$ while the number of rows of B_α is $rank(S_{\alpha-1})$.

So the number of rows in C_α is

$$\begin{aligned} dim_{\mathbb{C}}\big(In_\alpha(\mathcal{I})\big) &= rank\,(\,S_\alpha\,) - rank\,(\,S_{\alpha-1}\,) \\ &= \left(\binom{\alpha+s}{s} - nullity\,(\,S_\alpha\,)\right) - \left(\binom{\alpha-1+s}{s} - nullity\,(\,S_{\alpha-1}\,)\right) \\ &= \binom{\alpha-1+s}{s-1} - \mathrm{h}(\alpha) \qquad (11) \end{aligned}$$

where the difference of binomial coefficients comes from either Pascal's identity or noting that the number of monomials of total degree less than or equal to α minus those of total degree less than or equal to $\alpha - 1$ is the number of monomials of total degree exactly α. We also applied identity (6). The lemma follows by combining equations (9) and (11). □

Remarks: From an algebraic-geometric point of view the rings $\mathbb{C}\{x_1,\ldots,x_s\}$ and $\mathcal{A} = \mathbb{C}\{x_1,\ldots,x_s\}/F\mathbb{C}\{x_1,\ldots,x_s\}$ are *local rings* with $\mathfrak{M}^1$ and $\mathfrak{m}^1$ as their respective unique maximal ideals. The ideals $\mathfrak{M}^\alpha$ and $\mathfrak{m}^\alpha$ are powers of these maximal ideals. $\mathrm{Gr}(\mathcal{A}) = \bigoplus \mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}$ is known as the *associated graded ring* (see [6, 11, 7]) also known as the *tangent cone*. $\mathrm{Gr}(\mathcal{A})$ is a *standard graded ring*, these have a Hilbert function $\mathrm{H}(\alpha)$ defined as in (8). Much is known about the behavior of such Hilbert functions [6, 7, 9]. For example, if for some fixed β one has $H(\beta) = \gamma \le \beta$ then $H(\alpha) \le \gamma$ for all $\alpha \ge \beta$. We will start the next section with a proof of this in the case $\gamma = 0$.

Remark on positive dimension: Lemma 4 does not require $\hat{\mathbf{x}}$ to be an isolated zero. In general there is a polynomial $HP(\alpha)$ so that $H(\alpha) = HP(\alpha)$ for $\alpha \gg 0$. HP gives information about the local structure of the zero set $V(\mathcal{I})$ near $\hat{\mathbf{x}}$. For example, the degree of $HP(\alpha)$ is one less than the local dimension of $V(\mathcal{I})$ and the leading coefficient of HP is the local degree. Again see [6, 7] for more details.

Example 1 Consider the system $F = \{\sin(x-y) + x^3, x - y + sin(y)^3\}$. We calculate S_4 and reduce this exact matrix to reduced row echelon form, column headings provided for the reader's convience.

1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	y^3	x^4	x^3y	x^2y^2	xy^3	y^4
0	1	−1	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	1	0	−1	0	0	0	0	0	0	0	0	2
0	0	0	0	1	−1	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	1	0	0	−1	0	0	0	0	0
0	0	0	0	0	0	0	1	0	−1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	−1
0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1
0	0	0	0	0	0	0	0	0	0	0	0	1	0	−1
0	0	0	0	0	0	0	0	0	0	0	0	0	1	−1

The boxes then show the matrices C_α corresponding to the inital forms of degrees 1,2,3 and 4. So, for example, that the initial forms of degree 3 are spanned by $x^3 - y^3, x^2y - y^3$ and $xy^2 - y^3$.

In each case these initial form matrices are not full rank, each has one more column than row. This tells us that $\mathrm{h}(1) = \mathrm{h}(2) = \mathrm{h}(3) = \mathrm{h}(4) = 1$. Always $\mathrm{h}(0) = 1$. So knowing only the reduced row echelon form of S_4 tells us the multiplicity is at least 5.

Finding the reduced row echelon form of S_5 shows the above matrix in the upper 10×15 block and the identity matrix in the lower right 6×6 block and zeros elsewhere so we then know $\mathrm{h}(5) = 0$. In the next section we will see that the row reduction of S_5 tells us everything we need to know about the multiplicity structure of this system. $\square$

3.1 The Local Finiteness, Depth and Multiplicity Consistency Theorems.

We need one more lemma which is a special case of Nakayama's Lemma [1, 6].

Lemma 5 *Assume* $F = \{f_1, \ldots, f_t\}$ *is a system of functions that are analyitic in a neighborhood of their common zero* $\hat{\mathbf{x}} = \mathbf{0}$ *in* $\mathbb{C}^s$. *Let* $\mathcal{I} = F\mathbb{C}\{x_1, \ldots, x_s\}$, $\mathcal{A} = \mathbb{C}\{x_1, \ldots, x_s\}/\mathcal{I}$, $\mathfrak{M}^\alpha$ *the ideal of series of order* α *or greater and there is a filtration*

$$\mathcal{A} \quad = \quad \mathfrak{m}^0 \supseteq \mathfrak{m}^1 \supseteq \mathfrak{m}^2 \supseteq \ldots \tag{12}$$

given by $\mathfrak{m}^\alpha = \mathfrak{M}^\alpha/(\mathfrak{M}^\alpha \cap \mathcal{I})$. *Let* $\mathrm{h}(\alpha)$ *be one of the equivalent Hilbert functions of the previous section. Then*

(i) *if for some* $\beta > 0$, $\mathfrak{m}^\beta = \mathfrak{m}^{\beta+1}$ *then* $\mathfrak{M}^\beta \subseteq \mathcal{I}$ *and*

(ii) *if* $\mathrm{h}(\beta) = 0$ *then* $\mathrm{h}(\alpha) = 0$ *for all* $\alpha \geq \beta$.

Proof. By Lemma 2 $\mathfrak{m}^\beta$, a quotient of $\mathfrak{M}^\beta$, is generated by monomials of total degree β in $x_1, \ldots, x_s$. For convenience list these monomials in some order $a_1, \ldots, a_n$ where $n = \binom{\beta+s-1}{s-1}$. Likewise $\mathfrak{m}^{\beta+1}$ is generated by monomials of total degree $\beta + 1$, that is, monomials of the non-unique form $x_i a_j$. But assuming $\mathfrak{m}^\beta = \mathfrak{m}^{\beta+1}$ then each a_j is a sum of elements of $\mathbb{C}\{x_1, \ldots, x_s\}$ times the $x_i a_j$. Collecting the a_j we have that

$$a_k \quad = \quad c_{k,1}a_1 + c_{k,2}a_2 + \cdots + c_{k,n}a_n$$

in the ring $\mathcal{A}$. Note that each $c_{k,j}$ is represented by an element of $\mathfrak{m}^1$. We then get a system of equations in $\mathcal{A}$

$$\begin{aligned} 0 &= (c_{1,1} - 1)a_1 + c_{1,2}a_2 + \cdots + c_{1,n}a_n \\ 0 &= c_{2,1}a_1 + (c_{2,2} - 1)a_2 + \cdots + c_{2,n}a_n \\ &\vdots \\ 0 &= c_{n,1}a_n + \cdots + c_{n,n-1}a_{n-1} + (c_{n,n} - 1)a_n. \end{aligned}$$

But each $(c_{\alpha,\alpha} - 1)$ is invertible in $\mathbb{C}\{x_1, \ldots, x_s\}$ because it is a convergent series with constant term -1 and so has a reciprocal convergent on a small open set about $\mathbf{0}$. But then $(c_{\alpha,\alpha} - 1)$ is also invertible in the quotient ring $\mathcal{A}$. More generally any sum of an invertible element with an element of $\mathfrak{m}^1$ is invertible. Thus the determinant of the system is immediately seen to be invertible. It follows the system is non-singular so each $a_k = 0$ in $\mathcal{A}$ which means that a_k represents some element in $\mathcal{I}$. Thus $\mathfrak{M}^\beta \subseteq \mathcal{I}$ proving (i).

For (ii) note by Lemma 4 $\mathrm{h}(\beta) = \mathit{dim}_{\mathbb{C}}\left(\mathfrak{m}^\beta/\mathfrak{m}^{\beta+1}\right)$ so if $\mathrm{h}(\beta) = 0$ then $\mathfrak{m}^\beta = \mathfrak{m}^{\beta+1}$ so by (i) $\mathfrak{M}^\beta \subseteq \mathcal{I}$. But since we have a descending filtration, for any $\alpha \geq \beta$ $\mathfrak{M}^\alpha \subseteq \mathfrak{M}^\beta \subseteq \mathcal{I}$ so $\mathfrak{m}^\alpha = \mathfrak{M}^\alpha/(\mathfrak{M}^\alpha \cap \mathcal{I}) = \mathfrak{M}^\alpha/\mathfrak{M}^\alpha = 0$ in $\mathcal{A}$. Thus $\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1} = 0$ and so $\mathrm{h}(\alpha) = 0$ by Lemma 4.
$\square$

Theorem 1 (Local Finiteness Theorem) *Let $\mathcal{R}$ be the ring of analytic functions on open set $\mathcal{U} \subseteq \mathbb{C}^s$ and assume $\hat{\mathbf{x}} = \mathbf{0} \in \mathcal{U}$ is a zero of the sytem $F = \{f_1, \ldots, f_t\} \subset \mathcal{R}$. Let $\mathrm{h}(\alpha)$ be the hilbert function defined by equation (3). Then the following are equivalent:*

(i) *$\hat{\mathbf{x}}$ is an isolated zero of the system F.*

(ii) *The local ring $\mathcal{A} = \mathbb{C}\{x_1, \ldots, x_s\}/F\mathbb{C}\{x_1, \ldots, x_s\}$ is finite dimensional as a $\mathbb{C}$ vector space.*

(iii) $\sum_{\alpha \geq 0} \mathrm{h}(\alpha) < \infty$.

(iv) *$dim\big(\mathcal{D}_{\hat{\mathbf{x}}}(F)\big)$ is finite.*

(v) *For large α the Macaulay matrix S_α is row equivalent to a matrix $\begin{bmatrix} R & B \\ 0 & C \end{bmatrix}$ where C is the $n \times n$ identity matrix, $n = \binom{\alpha+s-1}{s-1}$.*

(vi) *For large α given any monomial $\mathbf{x}^{\mathbf{j}}$ of total degree α there is $p_{\mathbf{j}} \in \mathcal{R}$ with $ord(p_{\mathbf{j}}) > \alpha$ such that $\mathbf{x}^{\mathbf{j}} + p_{\mathbf{j}} \in F\mathbb{C}[x_1, \ldots, x_s]$.*

Proof. (i) $\Rightarrow$ (ii): By Lemma 1 for each $j \in \{1, \ldots, s\}$ there is an integer e_j with $x^{e_j} \in F\mathbb{C}\{x_1, \ldots, x_s\}$. 5 So if $\beta = \max\{e_j\}$ then any monomial $\mathbf{x}^{\mathbf{j}}$ of total degree β is contained in $F\mathbb{C}\{x_1, \ldots, x_s\}$, that is with the notation of Lemma 2 $\mathfrak{M}^\alpha \subseteq F\mathbb{C}\{x_1, \ldots, x_s\}$. Thus the filtration (12) terminates at or before β because $\mathfrak{m}^\beta = 0$. But then

$$\begin{aligned} dim_{\mathbb{C}}(\mathcal{A}) &= dim_{\mathbb{C}}(\mathcal{A}) - dim_{\mathbb{C}}(\mathfrak{m}^\beta) \\ &= \Big(dim_{\mathbb{C}}(\mathcal{A}) - dim_{\mathbb{C}}(\mathfrak{m}^1)\Big) + \Big(dim_{\mathbb{C}}(\mathfrak{m}^1) - dim_{\mathbb{C}}(\mathfrak{m}^2)\Big) + \cdots + \Big(dim_{\mathbb{C}}(\mathfrak{m}^{\beta-1}) - dim_{\mathbb{C}}(\mathfrak{m}^\beta)\Big) \\ &= dim_{\mathbb{C}}(\mathfrak{m}^0/\mathfrak{m}^1) + dim_{\mathbb{C}}(\mathfrak{m}^1/\mathfrak{m}^2) + \cdots + dim_{\mathbb{C}}(\mathfrak{m}^{\beta-1}/\mathfrak{m}^\beta) < \infty \end{aligned} \quad (13)$$

since each $dim_{\mathbb{C}}(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}) < \binom{\alpha+s-1}{s-1}$.

(ii) $\Rightarrow$ (iii): Equation (13) without the last term shows that

$$dim_{\mathbb{C}}(\mathcal{A}) = \sum_{\alpha \geq 0} dim_{\mathbb{C}}(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}) = \sum_{\alpha \geq 0} h(\alpha). \quad (14)$$

So assuming the right hand side of this last equation is finite, so must be the left hand side.

(iii) $\Leftrightarrow$ (iv): Since by equation (3) $dim(\mathcal{D}_{\hat{\mathbf{x}}}) = \sum_{\alpha \geq 0} \mathrm{h}(\alpha)$ conditions (iii) and (iv) are equivalent.

(iii) $\Rightarrow$ (v): For $\sum_{\alpha \geq 0} \mathrm{h}(\alpha) < \infty$ all but finitely many of the positive integers $\mathrm{h}(\alpha) = 0$, so $\mathrm{h}(\alpha) = 0$ for large α. 5 But then equation (11) says that $dim_{\mathbb{C}}\big(In_\alpha(\mathcal{I})\big) = \binom{\alpha+s-1}{s-1}$. But this says the matrix C_α in (10) is a square matrix with independent rows, so can be transformed to the identity matrix by row operations.

(v) $\Rightarrow$ (vi): By construction, since $\hat{\mathbf{x}} = \mathbf{0}$, Peach row of S_α consists of the coefficients of $jet(\mathbf{x}^{\mathbf{j}} f_i, \alpha)$ for some monomial in $x_1, \ldots, x_s$. In the row equivalent matrix rows are then linear combinations of such jets, but since the jet operator is linear the row with a 1 in the column indexed $\mathbf{j}$ consists of the coefficients of the jet of $\mathbf{x}^{\mathbf{j}} + p_{\mathbf{j}} \in F\mathbb{C}[x_1, \ldots, x_s]$ with $ord(p_{\mathbf{j}}) > \alpha$.

(vi) $\Rightarrow$ (i): In particular (vi) says that there exist α such that for each monomial of the form x_j^α there is $x_j^\alpha + p_j \in F\mathbb{C}[x_1, \ldots, x_s]$ so condition (iii) of Lemma 1 is satisfied and hence $\hat{\mathbf{x}}$ is an isolated zero. $\square$

Remark on Local Finiteness Theorem: This theorem should be compared with the *Finiteness Theorem* [1, p. 37] which essentially states that for an ideal $\mathcal{I}$ of $\mathbb{C}[x_1, \ldots, x_s]$, the following are equivalent:

a. $\mathbb{C}[x_1, \ldots, x_s]/\mathcal{I}$ is finite dimensional over $\mathbb{C}$.
b. $V(\mathcal{I})$ is finite.
c. For each $1 \leq i \leq s$ there is $m_i > 0$ so that there is an element of $\mathcal{I}$ of the form $p_i + x_i^{m_i}$ in $\mathcal{I}$ where the total degree of p_i is less than m_i.

The comparison of this theorem with our Theorem 1 highlights the difference between the local and global theories, they are practically opposite.

Theorem 2 (Depth Theorem) *Let* $F = \{f_1, \ldots, f_t\}$ *be a system of analytic functions in an open set of* $\mathbb{C}^s$ *at an isolated zero* $\hat{\mathbf{x}} = \mathbf{0}$. *Then there is a number* $\delta = \delta_{\hat{\mathbf{x}}}(F)$ *called the* depth of the isolated zero $\hat{\mathbf{x}}$ *satisfying the following equivalent conditions.*

(i) δ *is the largest integer with* $\mathrm{h}(\delta) \neq 0$

(ii) δ *is the smallest integer with* $\mathrm{h}(\delta + 1) = 0$.

(iii) δ *is the highest differential order of a functional in* $\mathcal{D}_{\hat{\mathbf{x}}}(F)$.

(iv) δ *is the smallest integer so that the Macaulay matrix* $S_{\delta+1}$ *is row equivalent to a matrix* $\begin{bmatrix} R & B \\ 0 & C \end{bmatrix}$ *where* C *is the* $n \times n$ *identity matrix,* $n = \binom{\delta+s}{s-1}$.

Proof. (i) $\Leftrightarrow$ (ii): Just notice that by Lemma 5 once $\mathrm{h}(\alpha) = 0$ it remains zero.

(i) $\Leftrightarrow$ (iii): This is equation (3)

(i) $\Leftrightarrow$ (iv): This follows from equations (10) and (11). □

Theorem 3 (Consistency Theorem) *Suppose* $\hat{\mathbf{x}} = \mathbf{0}$ *is an isolated zero of the analytic system* $F = \{f_1, \ldots, f_t\}$ *defined on an open set in* $\mathbb{C}^s$. *If the system* F *is square,* $t = s$ *and analytic the* multiplicity of the system F at $\hat{\mathbf{x}}$ *is the number* m *satisfying the following equivalent conditions.*

(i) $m = \mathit{dim}_{\mathbb{C}}\Big(\mathbb{C}\{x_1, \ldots, x_s\}/F\mathbb{C}\{x_1, \ldots, x_s\}\Big)$.

(ii) $m = \sum_{\alpha \geq 0} \mathrm{h}(\alpha)$.

(iii) $m = \mathit{dim}\big(\mathcal{D}_{\hat{\mathbf{x}}}(F)\big)$.

(iv) $m = \mathit{nullity}\,(\,S_\alpha\,)$ *for* $\alpha \geq \delta_{\hat{\mathbf{x}}}(F)$.

(v) *A small perturbation of* F *will break* $\hat{\mathbf{x}}$ *into a cluster of* m *points, counted by multiplicity.*

In addition, if $F \subset \mathbb{C}[x_1, \ldots, x_s]$ *is a system of polynomials then the multiplicity agrees with the standard algebraic definition of local multiplicity. On the other hand, if it exists, the multiplicity of a non-linear system* F *is the multiplicity of the polynomial system* $\mathit{jet}(F, \delta + 1) = \{\mathit{jet}(f_1, \delta + 1), \ldots, \mathit{jet}(f_t, \delta+1)\}$ *where* $\delta = \delta_{\hat{\mathbf{x}}}(F)$ *is the depth of the isolated zero* $\hat{\mathbf{x}}$ *as defined in Definition 1.*

Proof. The equivalence of (i) and (ii) is just equation (14) while that of (ii) and (iii) is (3). The equivalence of (ii) and (iv) comes from equation (14) and Lemma 4.

Condition (v) says that for any functions $g_1, \ldots, g_t$ that are analytic in Ω, there is a $\theta > 0$ such that the perturbed system $F_\varepsilon = \{f_1 + \varepsilon g_1, \ldots, f_t + \varepsilon g_t\}$ has exactly m zeros in Ω counting multiplicities for all $0 < |\varepsilon| < \theta$. So (i) $\Rightarrow$ (v) by defining $\tilde{f}_i : \mathbb{C}^s \times \mathbb{C}^1 \to \mathbb{C}^1$ by $\tilde{f}_i(\mathbf{x}, \varepsilon) = f_i(\mathbf{x}) - \varepsilon g_i(\mathbf{x})$ and applying Lemma 3 which uses (i) as the definition of multiplicity. The converse (v) $\Rightarrow$ (i) follows from the previous sentence together with the existence and uniqueness of the multiplicity of an isolated zero of an analytic system.

In the polynomial case the local multiplicity is defined by

$$m = \mathit{dim}_{\mathbb{C}}\, \mathbb{C}[x_1, \ldots, x_s]_{\langle x_1, \ldots, x_s \rangle} / F\mathbb{C}[x_1, \ldots, x_s]_{\langle x_1, \ldots, x_s \rangle}$$

in [1] immediately following its Definition 2.1. This is equivalent to (i) above by [1, Prop. 2.11].

Definition 1 requires the system F to have derivatives up to order γ where $\mathcal{D}_{\hat{\mathbf{x}}}^{\gamma-1}(F) = \mathcal{D}_{\hat{\mathbf{x}}}^{\gamma}(F)$. Since the Macaulay matrix S_α of F is identical to that of the polynomial system $\mathit{jet}(F, \alpha)$ for $\alpha \leq \gamma$. By (iv) of Theorem 2 $\mathit{jet}(F, \delta + 1)$ has depth $\delta \leq \gamma - 1$ and so by (iii) above the multiplicity of the polynomial system $\mathit{jet}(F, \delta + 1)$ is $\mathit{dim}\big(\mathcal{D}_{\hat{\mathbf{x}}}(F)\big)$ equals the multiplicity of F defined in Definition 1. □

Remarks: The local, or algebraic, multiplicity is sometimes known as the *intersection multiplicity* and appears in either of the equivalent forms (i) or (ii) or other more abstract equivalents, some of which are called *arithmetic mutltiplicity.* See [5, p. 127] for a historical discussion of this concept. In commutative algebra the term *regularity index* or just *index* is used instead of our *depth.* Specifically the regularity index is the first β such that $\mathrm{H}(\beta) = \mathrm{HP}(\beta)$ for all $\alpha \geq \beta$ where HP is the Hilbert Polynomial [7]. In the case of an isolated zero the Hilbert Polynomial is identically zero so the regularity index of $\hat{\mathbf{x}}$ is $\delta_{\hat{\mathbf{x}}}(F) + 1$.

In [6, Prop 5.5.12] a calculation of the associated graded ring of

$$\mathbb{C}[x_1, \ldots, x_s]_{\langle x_1, \ldots, x_s \rangle} / F\mathbb{C}[x_1, \ldots, x_s]_{\langle x_1, \ldots, x_s \rangle}$$

in terms of the initial ideal is given which is essentially identical to our argument in the proof of Lemma 4. Equation (14) shows that the $\mathbb{C}$-dimension of a zero dimensional local ring is the same as the $\mathbb{C}$-dimension of its associated graded ring so we have essentially given a proof of this proposition.

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